

THE \mathcal{A} -TRUNCATED K -MOMENT PROBLEM

JIAWANG NIE

ABSTRACT. Let $\mathcal{A} \subseteq \mathbb{N}^n$ be a finite set, and $K \subseteq \mathbb{R}^n$ be a compact semialgebraic set. An \mathcal{A} -truncated moment sequence (\mathcal{A} -tms) is a vector $y = (y_\alpha)$ indexed by elements in \mathcal{A} . The \mathcal{A} -truncated K -moment problem (\mathcal{A} -TKMP) studies whether or not a given \mathcal{A} -tms y admits a K -measure μ , i.e., a nonnegative Borel measure μ supported in K such that $y_\alpha = \int_K x^\alpha d\mu$ for all $\alpha \in \mathcal{A}$. This paper proposes a numerical algorithm for solving \mathcal{A} -TKMPs. It is based on finding a flat extension of y by solving a hierarchy of semidefinite relaxations $\{(\text{SDR})_k\}_{k=1}^\infty$ for a moment optimization problem, whose objective R is generated in a certain randomized way. If y admits no K -measures and $\mathbb{R}[x]_{\mathcal{A}}$ is K -full (there exists $p = \sum_{\alpha \in \mathcal{A}} p_\alpha x^\alpha$ that is positive on K), then $(\text{SDR})_k$ is infeasible for all k big enough, which gives a certificate for the nonexistence of representing measures. If y admits a K -measure, then for almost all generated R , we prove that: i) we can asymptotically get a flat extension of y by solving the hierarchy $\{(\text{SDR})_k\}_{k=1}^\infty$; ii) under a general condition that is almost sufficient and necessary, we can get a flat extension of y by solving $(\text{SDR})_k$ for some k ; this occurred in all our numerical experiments; iii) the obtained flat extensions admit a r -atomic K -measure with $r \leq |\mathcal{A}|$. The decomposition problems for completely positive matrices and sums of even powers of real linear forms, and the standard truncated K -moment problems, are special cases of \mathcal{A} -TKMPs. They can be solved numerically by this algorithm.

1. INTRODUCTION

Let $\mathcal{A} \subseteq \mathbb{N}^n$ be a finite set (\mathbb{N} is the set of nonnegative integers). An \mathcal{A} -truncated moment sequence (\mathcal{A} -tms) is a vector $y = (y_\alpha)_{\alpha \in \mathcal{A}}$ in $\mathbb{R}^{\mathcal{A}}$ (the space of real vectors indexed by elements in \mathcal{A}). For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, denote $|\alpha| := \alpha_1 + \dots + \alpha_n$. The degree of \mathcal{A} is $\deg(\mathcal{A}) := \max\{|\alpha| : \alpha \in \mathcal{A}\}$. Let K be the semialgebraic set

$$(1.1) \quad K := \{x \in \mathbb{R}^n : h(x) = 0, g(x) \geq 0\}$$

defined by two tuples of polynomials $h = (h_1, \dots, h_{m_1})$ and $g = (g_1, \dots, g_{m_2})$. A nonnegative Borel measure μ on \mathbb{R}^n is called a K -measure if its support, denoted by $\text{supp}(\mu)$, is contained in K . For $x := (x_1, \dots, x_n) \in \mathbb{R}^n$ and $\alpha := (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, denote $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. An \mathcal{A} -tms y is said to admit a K -measure μ if

$$y_\alpha = \int_K x^\alpha d\mu \quad \forall \alpha \in \mathcal{A}.$$

The measure μ satisfying the above is called a K -representing measure for y . Let $\text{meas}(y, K)$ denote the set of all K -measures admitted by y . Denote

$$\mathcal{R}_{\mathcal{A}}(K) := \{y \in \mathbb{R}^{\mathcal{A}} : \text{meas}(y, K) \neq \emptyset\}.$$

1991 *Mathematics Subject Classification.* 44A60, 47A57, 90C22, 90C90.

Key words and phrases. \mathcal{A} -truncated moment sequence, \mathcal{A} -truncated K -moment problem, completely positive matrices, flat extension, moment matrix, localizing matrix, representing measure, semidefinite program, sums of even powers.

The \mathcal{A} -truncated K -moment problem (\mathcal{A} -TKMP) is to determine whether a given \mathcal{A} -tms y admits a K -measure or not? If it does not, can we get a certificate for that? If it does, how can we obtain a K -representing measure? Preferably, we are often interested in finitely atomic K -representing measures. (A measure is *finitely atomic* if its support is a finite set, and is *r-atomic* if its support consists of at most r distinct points.) This paper is to present a numerical algorithm for solving \mathcal{A} -TKMPs.

1.1. Two special cases. Many interesting computational problems can be formulated as \mathcal{A} -TKMPs with appropriate \mathcal{A} and K . Here we list two of them.

The first one is the decomposition problem for *completely positive* matrices (cf. [5]). A symmetric $n \times n$ matrix C is called *completely positive* if there exist vectors $u_1, \dots, u_r \in \mathbb{R}_+^n$ (the nonnegative orthant of \mathbb{R}^n) such that

$$C = u_1 u_1^T + \dots + u_r u_r^T.$$

The above is called a *CP-decomposition* of C , if it exists. How can we determine whether a matrix C is completely positive or not? If it is not, can we get a certificate for that? If it is, how can we get a CP-decomposition for C ? As we will see in Section 6, this problem can be formulated as an \mathcal{A} -TKMP with

$$\mathcal{A} = \{\alpha \in \mathbb{N}^n : |\alpha| = 2\}, \quad K = \{x \in \mathbb{R}_+^n : x_1 + \dots + x_n = 1\}.$$

The second one is the decomposition problem for *sums of even powers* (SOEP) of real linear forms (cf. [35]). (A form is a homogeneous polynomial.) A form f of an even degree m is SOEP if there exist real linear forms L_1, \dots, L_r such that

$$f = L_1^m + \dots + L_r^m.$$

The above is called an *SOEP-decomposition* of f , if it exists. How can we determine whether a form f is SOEP or not? If it is not, can we get a certificate for that? If it is, how can we get an SOEP-decomposition for f ? As we will see in Section 6, this problem can also be formulated as an \mathcal{A} -TKMP with

$$\mathcal{A} = \{\alpha \in \mathbb{N}^n : |\alpha| = m\}, \quad K = \{x \in \mathbb{R}^n : x^T x = 1, x_1 + \dots + x_n \geq 0\}.$$

It is typically quite difficult to detect the existence of or computing CP/SOEP-decompositions. In the prior existing work, there are no much efficient numerical methods for solving such decomposition problems (except some special cases), in the author's best knowledge. In this paper, we will show how these problems can be solved numerically as special cases of \mathcal{A} -TKMPs.

1.2. Standard truncated K -moment problems. Denote $\mathbb{N}_d^n := \{\alpha \in \mathbb{N}^n : |\alpha| \leq d\}$. When $\mathcal{A} = \mathbb{N}_d^n$, the \mathcal{A} -TKMP is specialized to the standard truncated K -moment problem (TKMP), and an \mathbb{N}_d^n -tms is called a tms of degree d . Curto and Fialkow originally studied TKMPs and have made foundational work in the field. We refer to [9, 11, 12] and the references therein. Here we give a short review for TKMP.

Every tms $z \in \mathbb{R}^{\mathbb{N}_d^n}$ defines a *Riesz functional* \mathcal{L}_z acting on $\mathbb{R}[x]_d$ (the space of real polynomials in $x := (x_1, \dots, x_n)$ of degrees at most d) as

$$(1.2) \quad \mathcal{L}_z \left(\sum_{|\alpha| \leq d} p_\alpha x^\alpha \right) := \sum_{|\alpha| \leq d} p_\alpha z_\alpha.$$

We also denote $\langle p, z \rangle := \mathcal{L}_z(p)$. The Riesz functional \mathcal{L}_z is K -positive if

$$\mathcal{L}_z(p) \geq 0 \quad \forall p \in \mathbb{R}[x]_d : p|_K \geq 0.$$

The K -positivity of \mathcal{L}_z is necessary for z to admit a K -measure. When K is compact, it is also sufficient, which can be implied from the proof of Tchakaloff's Theorem [39]. However, it is typically very difficult to check whether a Riesz functional is K -positive or not.

A more favorable condition than K -positivity of \mathcal{L}_z is *flatness*. For convenience of description, suppose $K = \mathbb{R}^n$. We denote $X \succeq 0$ (resp., $X \succ 0$) if the matrix X is symmetric positive semidefinite (resp., definite). For a tms $z \in \mathbb{R}^{\mathbb{N}_{2k}^n}$, define $M_k(z)$ to be the symmetric matrix, which is linear in z , such that

$$\mathcal{L}_z(p^2) = p^T M_k(z) p \quad \forall p \in \mathbb{R}[x]_k.$$

(For convenience, we also use p to denote the vector of coefficients of $p(x)$ in the graded lexicographical ordering.) The matrix $M_k(z)$ is called a k -th order moment matrix. If z admits a K -measure μ , then

$$p^T M_k(z) p = \mathcal{L}_z(p^2) = \int_K p^2 d\mu \geq 0 \quad \forall p \in \mathbb{R}[x]_k.$$

This implies that

$$(1.3) \quad M_k(z) \succeq 0.$$

Hence, (1.3) is necessary for z to admit a measure on \mathbb{R}^n , but typically not sufficient. However, if (1.3) is satisfied and z is flat, i.e.,

$$(1.4) \quad \text{rank } M_{k-1}(z) = \text{rank } M_k(z),$$

then z admits a unique measure, which is r -atomic with $r = \text{rank } M_k(z)$. When K is a semialgebraic set as in (1.1), there is a similar version of this result (cf. Theorem 2.2). This is an important result of Curto and Fialkow (cf. [11]). For convenience of notion, when $K = \mathbb{R}^n$, we simply say z is *flat* if it satisfies (1.4) and (1.3). (When K is as in (1.1), we say z is flat if it satisfies (2.4) and (2.2).)

Flatness is very useful for solving truncated moment problems. Let $y \in \mathbb{R}^{\mathbb{N}_d^n}$, $z \in \mathbb{R}^{\mathbb{N}_e^n}$ be two tms'. We say y is a *truncation* of z , or equivalently, z is an *extension* of y , if $d \leq e$ and $y_\alpha = z_\alpha$ for all $\alpha \in \mathbb{N}_d^n$. We denote by $z|_d$ the subvector of z , whose entries are indexed by $\alpha \in \mathbb{N}_d^n$. So, z is an extension of y if and only if $y = z|_d$. If z is flat and extends y , we say z is a *flat extension* of y . Similarly, if w is an extension of z and z is flat, then we say z is a *flat truncation* of w . Clearly, if y is not flat but has a flat extension w , then y and w commonly admit a finitely atomic measure (cf. Theorem 2.2). Indeed, Curto and Fialkow [11] further proved that: a tms $y \in \mathbb{R}^{\mathbb{N}_d^n}$ admits a K -measure if and only if it has a flat extension $w \in \mathbb{R}^{\mathbb{N}_{2k}^n}$ (cf. Theorem 2.3). However, little is known on how to get flat extensions.

When K is compact as in (1.1), Helton and Nie [20] proposed a semidefinite approach for solving TKMPs. Its basic idea is to find flat extensions through semidefinite relaxations with moment and localizing matrices. They proved that: if $y \in \mathbb{R}^{\mathbb{N}_d^n}$ admits no K -measures, then a semidefinite program will be infeasible; if $y \in \mathbb{R}^{\mathbb{N}_d^n}$ admits a K -measure and there exists $\mu \in \text{meas}(y, K)$ such that $\text{supp}(\mu) \subseteq \mathcal{Z}(p)$ ($\mathcal{Z}(p)$ denotes the real zero set of p), where p is a polynomial having certain weighted SOS type representations and $\mathcal{Z}(p)$ is finite, then a flat extension of y can be found by solving a semidefinite program. However, in more general cases, no much were known on how to get flat extensions.

1.3. Contributions. This paper studies the new and broader class of moment problems: \mathcal{A} -TKMPs. The CP/SOEP-decomposition problems and standard truncated K -moment problems are special cases of \mathcal{A} -TKMPs.

For an \mathcal{A} -tms y , a tms $w \in \mathbb{R}^{N_{2k}^n}$ is an extension of y (or y is a truncation of w) if $w_\alpha = y_\alpha$ for all $\alpha \in \mathcal{A}$. Denote by $w|_{\mathcal{A}}$ the subvector of w whose indices belong to \mathcal{A} . Clearly, if $w|_{\mathcal{A}} = y$ and w is flat, then w and y commonly admit a finitely atomic measure. When does y admit a K -measure? If it does, how can we get such a measure? If it does not, can we get a certificate for the nonexistence of such a measure? These are the main questions in \mathcal{A} -TKMPs.

This paper proposes a numerical algorithm (i.e., Algorithm 4.2) for solving \mathcal{A} -TKMPs. It is based on solving a hierarchy of semidefinite relaxations (we denote by $\{(\text{SDR})_k\}_{k=1}^\infty$ here for convenience), for a moment optimization problem whose objective is a Riesz functional $\langle R, w \rangle := \mathcal{L}_w(R)$. The objective R is generated in a certain randomized way. Assume K is given as in (1.1) and is compact. Denote $\mathbb{R}[x]_{\mathcal{A}} := \text{span}\{x^\alpha : \alpha \in \mathcal{A}\}$. If y admits no K -measures and $\mathbb{R}[x]_{\mathcal{A}}$ is K -full (i.e., there exists $p \in \mathbb{R}[x]_{\mathcal{A}}$ that is positive on K), then $(\text{SDR})_k$ will be infeasible for all k big enough. This gives a certificate for the nonexistence of a K -representing measure for y . If y admits a K -measure, then, for *almost all* generated R , this algorithm has the following properties:

- i) We can asymptotically get a flat extension of y by solving the hierarchy $\{(\text{SDR})_k\}_{k=1}^\infty$. So, the convergence is guaranteed with probability one.
- ii) We can get a flat extension of y by solving $(\text{SDR})_k$ for some k , under a general condition that is almost sufficient and necessary. This implies that the finite convergence is very likely to happen.
- iii) The obtained flat extensions admit a r -atomic K -measure with $r \leq |\mathcal{A}|$.

In all our numerical experiments, the finite convergence was always observed, and we got r -atomic K -representing measures with $r \leq |\mathcal{A}|$ when they exist.

CP/SOEP-decomposition problems are special cases of \mathcal{A} -TKMPs. So, this algorithm can be applied to solve them. If such decompositions do not exist, then the resulting $(\text{SDR})_k$ will be infeasible for all big k , which gives a certificate for the nonexistence; if they exist, we can get such decompositions, either asymptotically or in finitely many steps (very likely to happen). This algorithm can also solve standard TKMPs. In the author's best knowledge, this is the first numerical algorithm that has the aforementioned properties.

The paper is organized as follows. Section 2 reviews some backgrounds; Section 3 presents some properties of \mathcal{A} -TKMPs; Section 4 describes the algorithm; Section 5 proves the convergence results; Section 6 gives applications in solving CP/SOEP-decomposition problems and the standard TKMPs.

2. BACKGROUNDS

Notation The symbol \mathbb{N} (resp., \mathbb{R}) denotes the set of nonnegative integers (resp., real numbers). For $t \in \mathbb{R}$, $\lceil t \rceil$ (resp., $\lfloor t \rfloor$) denotes the smallest integer not smaller (resp., the largest integer not greater) than t . For $x \in \mathbb{R}^n$, denote by $[x]_{\mathcal{A}} = (x^\alpha)_{\alpha \in \mathcal{A}}$ the vector of monomials, whose exponents are from \mathcal{A} , ordered in the lexicographical ordering. Denote $\mathbb{N}_d^n := \{\alpha \in \mathbb{N}^n : |\alpha| \leq d\}$. When $\mathcal{A} = \mathbb{N}_d^n$, the vector $[x]_{\mathcal{A}}$ is denoted by $[x]_d$. Denote $[k] := \{1, \dots, k\}$. The symbol $\mathbb{R}[x] = \mathbb{R}[x_1, \dots, x_n]$ denotes the ring of polynomials in $x := (x_1, \dots, x_n)$ with real coefficients.

For a set $S \subseteq \mathbb{R}^n$, $|S|$ denotes its cardinality, and $\text{int}(S)$ denotes its interior. The symbol $\mathcal{P}_d(S)$ denotes the set of polynomials in $\mathbb{R}[x]_d$ that are nonnegative on S . The superscript T denotes the transpose of a matrix. For $u \in \mathbb{R}^n$, denote $\|u\|_2 := \sqrt{u^T u}$ and $B(u, r) := \{x \in \mathbb{R}^n : \|x - u\|_2 \leq r\}$. For a polynomial $p \in \mathbb{R}[x]$, $\|p\|_2$ denotes the 2-norm of the coefficient vector of p . Denote by $\mathbb{S}^k = \{x \in \mathbb{R}^{k+1} : \|x\|_2 = 1\}$ the k -dimensional unit sphere. Denote by \mathcal{S}^N the space of $N \times N$ real symmetric matrices. For a matrix A , $\|A\|_2$ denotes its standard operator 2-norm, and $\|A\|_F$ denotes the Frobenius norm of A , i.e., $\|A\|_F = \sqrt{\text{Trace}(A^T A)}$.

2.1. Standard truncated K -moment problems. Let $z \in \mathbb{R}^{N_d^n}$ and $K \subseteq \mathbb{R}^n$. Bayer and Teichmann [1] proved that: z admits a K -measure μ if and only if it admits a r -atomic K -measure ν with $r \leq \binom{n+d}{d}$. A nice exposition for this result can be found in Laurent [27, Theorem 5.8]. When K is compact, we can characterize the existence of representing measures via Riesz functionals. This can be implied by the proof of Tchakaloff's Theorem [39].

Theorem 2.1 (Tchakaloff). *Let K be a compact set in \mathbb{R}^n . A tms $z \in \mathbb{R}^{N_d^n}$ admits a K -measure if and only if its Riesz functional \mathcal{L}_z is K -positive.*

When K is noncompact, the above might not be true. A stronger condition is that \mathcal{L}_y is *strictly K -positive*, i.e.,

$$\mathcal{L}_y(p) > 0 \quad \forall p \in \mathbb{R}[x]_d : p|_K \geq 0, p|_K \not\equiv 0.$$

When K is a determining set (i.e., $p \equiv 0$ whenever $p|_K \equiv 0$), if \mathcal{L}_z is strictly K -positive, then z admits a K -measure (cf. [17, Theorem 1.3]). Typically, checking K -positivity or strict K -positivity is quite difficult.

A more useful condition than K -positivity is the positive semidefiniteness of localizing matrices. For $z \in \mathbb{R}^{N_{2k}^n}$ and $h \in \mathbb{R}[x]_{2k}$, define $L_q^{(k)}(z)$ to be the symmetric matrix, which is linear in z , such that

$$(2.1) \quad \mathcal{L}_z(qp^2) = p^T \left(L_q^{(k)}(z) \right) p \quad \forall p \in \mathbb{R}[x]_{k - \lceil \deg(q)/2 \rceil}.$$

The matrix $L_q^{(k)}(z)$ is called the k -th *localizing matrix* of q generated by z . When $q = 1$, $L_q^{(k)}(z)$ coincides with the moment matrix $M_k(z)$.

Let K be as in (1.1) and $g_0 = 1$. If $z \in \mathbb{R}^{N_{2k}^n}$ admits a K -measure μ , then

$$(2.2) \quad L_{h_i}^{(k)}(z) = 0 \ (i = 1, \dots, m_1), \quad L_{g_j}^{(k)}(z) \succeq 0 \ (j = 0, 1, \dots, m_2).$$

This is because

$$\begin{aligned} p^T L_{h_i}^{(k)}(z) p &= \mathcal{L}_z(h_i p^2) = \int_K h_i p^2 d\mu = 0, \\ p^T L_{g_j}^{(k)}(z) p &= \mathcal{L}_z(g_j p^2) = \int_K g_j p^2 d\mu \geq 0, \end{aligned}$$

for all $p \in \mathbb{R}[x]$ with $\max_{i,j} \{\deg(h_i p^2), \deg(g_j p^2)\} \leq 2k$. Therefore, (2.2) is necessary for z to admit a K -measure. Typically, it is not sufficient. Let

$$(2.3) \quad d_K := \max_{i \in [m_1], j \in [m_2]} \{1, \lceil \deg(h_i)/2 \rceil, \lceil \deg(g_j)/2 \rceil\}.$$

If, in addition to (2.2), z satisfies the rank condition

$$(2.4) \quad \text{rank } M_{k-d_K}(z) = \text{rank } M_k(z),$$

then z admits a unique K -measure, which is finitely atomic. This is an important result of Curto and Fialkow. For convenience of notion, we simply say z is flat if z satisfies (2.2) and (2.4).

Theorem 2.2 ([11]). *Let K be defined in (1.1). If a tms $z \in \mathbb{R}^{\mathbb{N}_{2k}^n}$ is flat (i.e., (2.4) and (2.2) hold), then z admits a unique K -measure, which is $\text{rank } M_k(z)$ -atomic.*

Flatness is very useful for solving truncated moment problems, as shown by Curto and Fialkow [9, 10]. A nice exposition for this can also be found in Laurent [26]. For a flat tms, its finitely atomic representing measure can be found by solving some eigenvalue problems, as shown by Henrion and Lasserre [21].

Clearly, if a tms y is not flat but has a flat extension z , then y admits a K -measure. Thus, the existence of a K -representing measure for y can be determined by investigating whether y has a flat extension or not. Indeed, Curto and Fialkow [11] proved the following important result.

Theorem 2.3 ([11]). *Let K be as in (1.1). Then a tms $y \in \mathbb{R}^{\mathbb{N}_d^n}$ admits a K -measure if and only if it has a flat extension $z \in \mathbb{R}^{\mathbb{N}_{2k}^n}$.*

There are other necessary conditions for admitting K -representing measures, like the recursively generated relation. We refer to Curto and Fialkow [8, 9, 10].

2.2. Ideals, varieties and positive polynomials. A subset $I \subseteq \mathbb{R}[x]$ is called an *ideal* if $I + I \subseteq I$ and $p \cdot q \in I$ for all $p \in I$ and $q \in \mathbb{R}[x]$. For a tuple $p = (p_1, \dots, p_m)$ of polynomials in $\mathbb{R}[x]$, denote by $I(p)$ the ideal generated by p_1, \dots, p_m , which is the set $p_1\mathbb{R}[x] + \dots + p_m\mathbb{R}[x]$. A variety is a subset of \mathbb{C}^n that consists of common zeros of a set of polynomials. A real variety is the intersection of a variety with \mathbb{R}^n . Given a tuple $p = (p_1, \dots, p_m)$, denote

$$V_{\mathbb{C}}(p) := \{v \in \mathbb{C}^n : p(v) = 0\}, \quad V_{\mathbb{R}}(p) := \{v \in \mathbb{R}^n : p(v) = 0\}.$$

A polynomial $f \in \mathbb{R}[x]$ is a *sum of squares* (SOS) if there exist $f_1, \dots, f_k \in \mathbb{R}[x]$ such that $f = f_1^2 + \dots + f_k^2$. The set of all SOS polynomials in n variables and of degree d is denoted by $\Sigma_{n,d}$. It is a convex cone in $\mathbb{R}^{\mathbb{N}_d^n}$ and has nonempty interior for any even $d > 0$. We refer to Reznick [34] for a survey on SOS polynomials.

Let K be as in (1.1), defined by the polynomial tuples $h = (h_1, \dots, h_{m_1})$ and $g = (g_1, \dots, g_{m_2})$. Denote $g_0 = 1$ and

$$(2.5) \quad I_{2k}(h) := \left\{ \sum_{i=1}^{m_1} h_i \phi_i \mid \text{each } \deg(h_i \phi_i) \leq 2k \right\},$$

$$(2.6) \quad Q_k(g) := \left\{ \sum_{i=0}^{m_2} g_i \sigma_i \mid \begin{array}{l} \text{each } \deg(\sigma_i g_i) \leq 2k \\ \text{and } \sigma_i \text{ is SOS} \end{array} \right\}.$$

Clearly, $I(h)$ is the union of all $I_{2k}(h)$, and $I_{2k}(h)$ is a truncation of $I(h)$ with degree $2k$. The union of all sets $Q_k(g)$, denoted by $Q(g)$, is called the *quadratic module* generated by g , and each $Q_k(g)$ is a truncation of $Q(g)$ with degree $2k$.

Clearly, if $f \in I(h) + Q(g)$, then f is nonnegative on K . The archimedean condition for (h, g) is that there exists $N > 0$ such that $N - \|x\|_2^2 \in I(h) + Q(g)$. When K is compact, the archimedean condition can be forced to be held by adding a redundant condition like $N - \|x\|_2^2 \geq 0$ to the description of K .

Theorem 2.4 (Putinar, [29]). *Let K be as in (1.1). Suppose the archimedean condition holds for (h, g) . If $f \in \mathbb{R}[x]$ is positive on K , then $f \in I(h) + Q(g)$.*

2.3. Semidefinite optimization with moment variables. Semidefinite programming (SDP) is very useful in solving moment problems. We refer to [40] for a survey on SDP, and refer to [25, 27, 20] for semidefinite programs arising from moment problems.

Let \mathcal{S}_+^N be the cone of positive semidefinite matrices in \mathcal{S}^N . Let $\mathcal{K} = \mathcal{S}_+^{N_1} \times \cdots \times \mathcal{S}_+^{N_\ell}$ be a cone in the space $\mathcal{S}^{N_1} \times \cdots \times \mathcal{S}^{N_\ell}$. A tuple $X = (X_1, \dots, X_\ell)$ belongs to \mathcal{K} if and only if each $X_i \in \mathcal{S}_+^{N_i}$. A general semidefinite program is

$$(2.7) \quad \min \quad \text{Trace}(CX) \quad \text{s.t.} \quad \mathcal{F}(X) = f, X \in \mathcal{K}$$

with C a constant tuple in $\mathcal{S}^{N_1} \times \cdots \times \mathcal{S}^{N_\ell}$, \mathcal{F} a linear operator from $\mathcal{S}^{N_1} \times \cdots \times \mathcal{S}^{N_\ell}$ to a vector space \mathbb{R}^m , and f a given vector in \mathbb{R}^m . Let \mathcal{F}^* be the adjoint operator of \mathcal{F} . The dual optimization problem of (2.7) is

$$(2.8) \quad \max \quad f^T y \quad \text{s.t.} \quad \mathcal{F}^*(y) + Z = C, Z \in \mathcal{K}.$$

Let h and g be the tuples of polynomials describing K as in (1.1). Denote

$$(2.9) \quad \Phi_k(g) := \left\{ w \in \mathbb{R}^{\mathbb{N}_{2k}^n} \mid L_{g_j}^{(k)}(w) \succeq 0, \quad j = 0, 1, \dots, m_2 \right\},$$

$$(2.10) \quad E_k(h) := \left\{ w \in \mathbb{R}^{\mathbb{N}_{2k}^n} \mid L_{h_i}^{(k)}(w) = 0, \quad i = 1, \dots, m_1 \right\}.$$

It can be shown that $\Phi_k(g)$ is the dual cone of $Q_k(g)$ and $E_k(h)$ is the dual cone of $I_{2k}(h)$ (cf. [25, 27]). Indeed, $E_k(h)$ is a subspace of $\mathbb{R}^{\mathbb{N}_{2k}^n}$.

A typical semidefinite program in this paper is

$$(2.11) \quad \min_w \quad c^T w \quad \text{s.t.} \quad w|_{\mathcal{A}} = y, \quad w \in \Phi_k(g) \cap E_k(h)$$

with given $c \in \mathbb{R}^{\mathbb{N}_d^n}$ ($\deg(\mathcal{A}) < d \leq 2k$) and $y \in \mathbb{R}^{\mathcal{A}}$. The dual optimization problem of (2.11) is

$$(2.12) \quad \max_{p \in \mathbb{R}[x]_{\mathcal{A}}} \quad \langle p, y \rangle \quad \text{s.t.} \quad c - p \in I_{2k}(h) + Q_k(g).$$

Any objective value of a feasible solution of (2.11) (resp., (2.12)) is an upper bound (resp., lower bound) for the optimal value of the other one (this is called *weak duality*). If one of them has an interior point (for (2.12) it means that there exists $p \in \mathbb{R}[x]_{\mathcal{A}}$ such that $c - p$ lies in the interior of $(Q_k(g) + I_{2k}(h)) \cap \mathbb{R}[x]_d$, and for (2.11) it means that there is a feasible w with $L_{g_j}^{(k)}(w) \succ 0$ for all j), then the other one has an optimizer and they have the same optimal value (this is called *strong duality*). We refer to [2, Section 2.4] for duality theory in linear conic optimization.

3. PROPERTIES OF \mathcal{A} -TKMPs

This section presents some properties of \mathcal{A} -truncated K -moment problems. They will be used in the latter sections.

3.1. \mathcal{A} -Riesz functionals. An \mathcal{A} -tms y defines an \mathcal{A} -Riesz functional \mathcal{L}_y that acts on $\mathbb{R}[x]_{\mathcal{A}}$ as

$$\mathcal{L}_y \left(\sum_{\alpha \in \mathcal{A}} p_{\alpha} x^{\alpha} \right) = \sum_{\alpha \in \mathcal{A}} p_{\alpha} y_{\alpha}.$$

The \mathcal{A} -Riesz functional \mathcal{L}_y is said to be *K-positive* if

$$\mathcal{L}_y(p) \geq 0 \quad \forall p \in \mathbb{R}[x]_{\mathcal{A}} : p|_K \geq 0,$$

and *strictly* K -positive if

$$\mathcal{L}_y(p) > 0 \quad \forall p \in \mathbb{R}[x]_{\mathcal{A}} : p|_K \geq 0, p|_K \not\equiv 0.$$

Clearly, if y admits a K -measure μ , then \mathcal{L}_y must be K -positive because

$$\mathcal{L}_y(p) = \int_K p d\mu \geq 0$$

whenever $p \in \mathbb{R}[x]_{\mathcal{A}}$ and $p|_K \geq 0$. So, the K -positivity of \mathcal{L}_y is a necessary condition for y to admit a K -measure. Indeed, it is also a sufficient if K is compact and $\mathbb{R}[x]_{\mathcal{A}}$ is K -full (i.e., there exists $p \in \mathbb{R}[x]_{\mathcal{A}}$ such that $p > 0$ on K). This is a result of Fialkow and Nie [18].

Theorem 3.1. ([18, Theorem 2.2]) *Suppose $K \subseteq \mathbb{R}^n$ is compact and $\mathbb{R}[x]_{\mathcal{A}}$ is K -full. Let y be an \mathcal{A} -tms such that \mathcal{L}_y is K -positive. Then, there exist $N \leq \dim \mathbb{R}[x]_{\mathcal{A}}$, $u_1, \dots, u_N \in K$, and $c_1, \dots, c_N > 0$, such that*

$$y = c_1[u_1]_{\mathcal{A}} + \dots + c_N[u_N]_{\mathcal{A}}.$$

The above theorem immediately implies the following.

Corollary 3.2. *Suppose $K \subseteq \mathbb{R}^n$ is compact and $\mathbb{R}[x]_{\mathcal{A}}$ is K -full. If an \mathcal{A} -tms y admits no K -measures, then there exists $p \in \mathbb{R}[x]_{\mathcal{A}}$ such that*

$$p|_K \geq 0, \quad \mathcal{L}_y(p) < 0.$$

There is a similar version of Theorem 2.3 for \mathcal{A} -TKMPs. Recall that a tms $w \in \mathbb{R}^{N_{2k}^n}$ is called flat if it satisfies (2.2) and (2.4).

Proposition 3.3. (i) *Let $K \subseteq \mathbb{R}^n$ be a set. Then, an \mathcal{A} -tms y admits a K -measure if and only if it admits a r -atomic K -measure with $r \leq |\mathcal{A}|$.*

(ii) *Let K be as in (1.1). Then, an \mathcal{A} -tms y admits a K -measure if and only if y is extendable to a flat tms $w \in \mathbb{R}^{N_{2k}^n}$ for some k .*

Proof. (i) The “if” direction is obvious. The “only if” direction can be proved by a formal repetition of the proof of Theorem 5.8 in Laurent [27]. The inequality $r \leq |\mathcal{A}|$ is implied by Carathéodory’s Theorem. Here we omit it for cleanness of presentation.

(ii) “ \Rightarrow ” Suppose y admits a measure on K . By item (i), it admits a r -atomic K -measure, say, $y = c_1[u_1]_{\mathcal{A}} + \dots + c_r[u_r]_{\mathcal{A}}$ with all $u_i \in K$, $c_i > 0$ and $r \leq |\mathcal{A}|$. For k big enough, the tms $w = c_1[u_1]_{2k} + \dots + c_r[u_r]_{2k} \in \mathbb{R}^{N_{2k}^n}$ is flat. Clearly, w is an extension of y .

“ \Leftarrow ” Suppose $w \in \mathbb{R}^{N_{2k}^n}$ is flat and $w|_{\mathcal{A}} = y$. By Theorem 2.2, w admits a K -measure μ and $w_{\alpha} = \int_K x^{\alpha} d\mu$ for all $\alpha \in \mathbb{N}_{2k}^n$. Since $w|_{\mathcal{A}} = y$, we have $y_{\alpha} = \int_K x^{\alpha} d\mu$ for all $\alpha \in \mathcal{A}$, i.e., y admits the K -measure μ . \square

3.2. Extremal extensions. For $d > \deg(\mathcal{A})$, define the d -th extension of y as

$$(3.1) \quad \mathcal{E}_d(y, K) = \{z \in \mathcal{R}_{\mathbb{N}_d^n}(K) : z|_{\mathcal{A}} = y\}.$$

Clearly, $\mathcal{E}_d(y, K)$ is convex. If $\text{meas}(y, K) \neq \emptyset$, then $\mathcal{E}_d(y, K) \neq \emptyset$.

Lemma 3.4. *Let $y \in \mathbb{R}^{\mathcal{A}}$. If K is compact and $\mathbb{R}[x]_{\mathcal{A}}$ is K -full, then $\mathcal{E}_d(y, K)$ is a compact convex set.*

Proof. The compactness of K implies $\mathcal{R}_{\mathbb{N}_d^n}(K)$ is closed, which can be implied by Theorem 2.1. So, $\mathcal{E}_d(y, K)$ is also closed. If $\text{meas}(y, K) = \emptyset$, then $\mathcal{E}_d(y, K) = \emptyset$ and we are done. Thus, we can assume $\text{meas}(y, K) \neq \emptyset$ and $\mathcal{E}_d(y, K) \neq \emptyset$. We need to prove that $\mathcal{E}_d(y, K)$ is bounded. Since $\mathbb{R}[x]_{\mathcal{A}}$ is K -full and K is compact, there exist $p \in \mathbb{R}[x]_{\mathcal{A}}$ and ϵ such that $p|_K \geq \epsilon > 0$. There exists $M > 0$ such that for all $x \in K$

$$-M \leq x^\alpha \leq M \quad \forall \alpha \in \mathbb{N}_d^n.$$

For all $z \in \mathcal{E}_d(y, K)$, the K -positivity of \mathcal{L}_z implies that

$$-Mz_0 \leq z_\alpha \leq Mz_0 \quad \forall \alpha \in \mathbb{N}_d^n.$$

Similarly, $p|_K \geq \epsilon$ implies $\mathcal{L}_y(p) \geq \epsilon z_0$. So,

$$|z_\alpha| \leq M\mathcal{L}_y(p)/\epsilon \quad \forall \alpha \in \mathbb{N}_d^n.$$

Hence, $\mathcal{E}_d(y, K)$ is bounded, which completes the proof. \square

Lemma 3.5. *Let $K \subseteq \mathbb{R}^n$, $d > \deg(\mathcal{A})$ and $y \in \mathcal{R}_{\mathcal{A}}(K)$. Suppose z is an extreme point of $\mathcal{E}_d(y, K)$. If $\mu \in \text{meas}(z, K)$, then μ must be r -atomic with $r \leq |\mathcal{A}|$.*

Proof. Choose an arbitrary $\mu \in \text{meas}(z, K)$. If μ is r -atomic and $r \leq |\mathcal{A}|$, then we are done. We derive a contradiction if either μ is r -atomic with $r > |\mathcal{A}|$, or μ is not finitely atomic.

First, consider the case that μ is r -atomic with $r > |\mathcal{A}|$, say,

$$\mu = c_1\delta(u_1) + \cdots + c_r\delta(u_r),$$

with distinct points $u_1, \dots, u_r \in K$ and $c_1, \dots, c_r > 0$. Here $\delta(v)$ denote the Dirac measure supported on the point v . For each $\alpha \in \mathbb{N}_d^n$, denote

$$w^{(\alpha)} := (u_1^\alpha, \dots, u_r^\alpha)^T \in \mathbb{R}^r.$$

We show that there must exist $\beta \in \mathbb{N}_d^n \setminus \mathcal{A}$ such that the system

$$(3.2) \quad \left(w^{(\alpha)}\right)^T t = 0 \quad (\forall \alpha \in \mathcal{A}), \quad \left(w^{(\beta)}\right)^T t \neq 0$$

has a solution $t = (t_1, \dots, t_r)$. Suppose otherwise, then for all $\beta \in \mathbb{N}_d^n \setminus \mathcal{A}$,

$$\left(w^{(\alpha)}\right)^T t = 0 \quad (\forall \alpha \in \mathcal{A}) \quad \implies \quad \left(w^{(\beta)}\right)^T t = 0.$$

This implies that, for every $\beta \in \mathbb{N}_d^n \setminus \mathcal{A}$, the vector $w^{(\beta)}$ is a linear combination of the vectors $w^{(\alpha)}$ ($\alpha \in \mathcal{A}$), i.e., there exist real numbers $p_{\beta, \alpha}$ such that

$$w^{(\beta)} = \sum_{\alpha \in \mathcal{A}} p_{\beta, \alpha} w^{(\alpha)}.$$

For each $\beta \in \mathbb{N}_d^n \setminus \mathcal{A}$, let

$$p_\beta := x^\beta - \sum_{\alpha \in \mathcal{A}} p_{\beta, \alpha} x^\alpha.$$

Note that $p_\beta(u_i) = 0$ for all $i = 1, \dots, r$. Let J be the ideal generated by p_β ($\beta \in \mathbb{N}_d^n \setminus \mathcal{A}$). Then J is zero-dimensional, and the dimension

$$D := \dim \mathbb{R}[x]/J \leq \dim \mathbb{R}[x]_{\mathcal{A}} = |\mathcal{A}|.$$

By Proposition 2.1 of Sturmfels [38], the number of common zeros of polynomials p_β ($\beta \in \mathbb{N}_d^n \setminus \mathcal{A}$) is equal to D , counting multiplicities. Hence, the cardinality of $V_{\mathbb{C}}(J)$ is at most $|\mathcal{A}|$. However, the distinct points u_1, \dots, u_r all belong to $V_{\mathbb{C}}(J)$

and $r > D$, which is a contradiction. So, there exists $t = (t_1, \dots, t_r)$ satisfying (3.2). Then, $(c_1, \dots, c_r) \pm \epsilon(t_1, \dots, t_r) > 0$ for $\epsilon > 0$ sufficiently small. Let

$$\mu_1 = \sum_{i=1}^r (c_i + \epsilon t_i) \delta(u_i), \quad \mu_2 = \sum_{i=1}^r (c_i - \epsilon t_i) \delta(u_i).$$

They are all nonnegative Borel measures supported in K . Let

$$z_1 = \int_K [x]_d d\mu_1, \quad z_2 = \int_K [x]_d d\mu_2.$$

Then, both z_1 and z_2 belong to $\mathcal{E}_d(y, K)$ and $z = \frac{1}{2}(z_1 + z_2)$. The inequality in (3.2) implies that $z_1 \neq z$, $z_2 \neq z$. This contradicts that z is extreme in $\mathcal{E}_d(y, K)$.

Second, consider the case that μ is not finitely atomic. Then $|\text{supp}(\mu)| = +\infty$. Choose $|\mathcal{A}| + 1$ distinct points, say, $v_1, \dots, v_{|\mathcal{A}|+1}$, from $\text{supp}(\mu)$. The support of μ is the smallest closed set S such that $\mu(\mathbb{R}^n \setminus S) = 0$ (cf. [27, Section 4]). So, there exists $\epsilon > 0$ such that $\mu(B(v_i, \epsilon)) > 0$ for all i and the balls $B(v_1, \epsilon), \dots, B(v_{|\mathcal{A}|+1}, \epsilon)$ are disjoint from each other. Let $T_i := B(v_i, \epsilon) \cap \text{supp}(\mu)$ for $i = 1, \dots, |\mathcal{A}|$, and

$$T_{|\mathcal{A}|+1} := \text{supp}(\mu) - \bigcup_{i=1}^{|\mathcal{A}|} B(v_i, \epsilon).$$

This results in the decomposition $\text{supp}(\mu) = T_1 \cup \dots \cup T_{|\mathcal{A}|+1}$. Note that $\mu(T_1) > 0, \dots, \mu(T_{|\mathcal{A}|+1}) > 0$ and $T_i \cap T_j = \emptyset$ whenever $i \neq j$. For each $j = 1, \dots, |\mathcal{A}| + 1$, let $\mu_j = \mu|_{T_j}$, the restriction of μ on T_j . Then

$$z = \int_K [x]_d d\mu_1 + \dots + \int_K [x]_d d\mu_{|\mathcal{A}|+1}.$$

Each tms $\int_K [x]_d d\mu_i$ admits a finitely atomic measure supported in T_i (cf. [27, Corollary 5.9]). Hence, there exists a measure $\theta \in \text{meas}(z, K)$ that is r -atomic with $r > |\mathcal{A}|$. Therefore, a contradiction can be obtained as in the first case.

Combining the above two cases, we know the conclusion is true. \square

3.3. Linear optimization over $\mathcal{E}_d(y, K)$. Let $d > \deg(\mathcal{A})$ and $R \in \mathbb{R}[x]_d$. Consider the linear moment optimization problem

$$(3.3) \quad \min_z \quad \langle R, z \rangle \quad \text{s.t.} \quad z|_{\mathcal{A}} = y, z \in \mathcal{R}_d(K).$$

The feasible set of (3.3) is $\mathcal{E}_d(y, K)$. Clearly, if z^* is the unique minimizer of (3.3), then z^* is an extreme point of $\mathcal{E}_d(y, K)$. Recall that $\mathcal{P}_d(K) = \{p \in \mathbb{R}[x]_d : p|_K \geq 0\}$. The dual optimization problem of (3.3) is

$$(3.4) \quad \max_{p \in \mathbb{R}[x]_{\mathcal{A}}} \quad \langle p, y \rangle \quad \text{s.t.} \quad R - p \in \mathcal{P}_d(K).$$

This is because the dual cone of $\mathcal{R}_d(K)$ is $\mathcal{P}_d(K)$, for any compact set K . This was shown by Tchakaloff [39] (also see Laurent [27, Section 5.2]).

Proposition 3.6. *Let $K \subseteq \mathbb{R}^n$ be a compact set and y be an \mathcal{A} -tms in $\mathcal{R}_{\mathcal{A}}(K)$.*

- (i) *If either $R|_K > 0$ or $\mathbb{R}[x]_{\mathcal{A}}$ is K -full, then (3.3) and (3.4) have the same optimal value and (3.3) has a minimizer.*
- (ii) *If \mathcal{L}_y is strictly K -positive, then (3.4) has a maximizer.*

Proof. (i) If $R|_K > 0$, then the origin is an interior point of (3.4). If $\mathbb{R}[x]_{\mathcal{A}}$ is K -full, then there exists $q \in \mathbb{R}[x]_{\mathcal{A}} \subseteq \mathbb{R}[x]_d$ that is strictly positive on K , so (3.4) also has an interior point. The problem (3.3) is feasible, because $y \in \mathcal{R}_{\mathcal{A}}(K)$. Thus, the strong duality holds and the conclusion is true (cf. [2, Section 2.4]).

(ii) From $y \in \mathcal{R}_{\mathcal{A}}(K)$, we know y has a flat extension $w \in \mathbb{R}^{\mathbb{N}_{2k}^n}$ for some k (cf. Proposition 3.3). Then the truncation $w|_d$ is a feasible point for (3.3). By weak duality, the optimal value of (3.4) is finite, say, η . Clearly, the feasible set of (3.4) is closed. Let $\{p_k\} \subseteq \mathbb{R}[x]_{\mathcal{A}}$ be a sequence such that each $R - p_k \in \mathcal{P}_d(K)$ and

$$\langle p_k, y \rangle \rightarrow \eta, \quad \text{as } k \rightarrow \infty.$$

Let $S_1 = \{f \in \mathbb{R}[x]_{\mathcal{A}} : f|_K \equiv 0\}$, and S_2 be the orthogonal complement of S_1 in $\mathbb{R}[x]_{\mathcal{A}}$. So, $\mathbb{R}[x]_{\mathcal{A}} = S_1 + S_2$, $S_1 \perp S_2$. Because $\langle f, y \rangle = 0$ and $f|_K \equiv 0$ for all $f \in S_1$, the p_k 's in the above can be chosen as $p_k \in S_2$ and $R - p_k \in \mathcal{P}_d(K)$.

If the sequence $\{p_k\}$ is bounded, then any of its accumulation points is a maximizer of (3.4) and we are done. Suppose otherwise $\{p_k\}$ is unbounded, say, $\|p_k\|_2 \rightarrow \infty$. Let $\hat{p}_k = p_k / \|p_k\|_2$ be the normalization. The sequence $\{\hat{p}_k\}$ is bounded, and we can generally assume $\hat{p}_k \rightarrow p^* \in S_2$. Clearly, $\|p^*\|_2 = 1$ and

$$\langle \hat{p}_k, y \rangle = \langle p_k, y \rangle / \|p_k\|_2 \rightarrow 0, \quad \text{as } k \rightarrow \infty.$$

This implies that $\langle p^*, y \rangle = 0$. From $(R - p_k) / \|p_k\|_2 \in \mathcal{P}_d(K)$, we get $-p^*|_K \geq 0$. Since $p^* \in S_2$ and $\|p^*\|_2 = 1$, we know $-p^*|_K \not\equiv 0$. The strict K -positivity of \mathcal{L}_y implies $\langle -p^*, y \rangle > 0$, which is a contradiction. Thus, the sequence $\{p_k\}$ must be bounded, and the proof is complete. \square

4. A SEMIDEFINITE ALGORITHM FOR \mathcal{A} -TKMPs

In this section, we present a numerical algorithm for solving \mathcal{A} -truncated K -moment problems. To determine whether an \mathcal{A} -tms y admits a K -measure or not, by Proposition 3.3, it is equivalent to investigating whether y has a flat extension or not. If it does not exist, then y does not admit a K -measure. If it exists, then we can get a finitely atomic representing measure for y through the flat extension.

The extension set $\mathcal{E}_d(y, K)$, defined in (3.1), is very useful in getting flat extensions. By Lemma 3.5, extreme points of $\mathcal{E}_d(y, K)$ admit only r -atomic K -measures with $r \leq |\mathcal{A}|$. Clearly, if (3.3) has a unique minimizer z^* , then z^* is an extreme point of $\mathcal{E}_d(y, K)$. A very useful fact is that when we minimize a generic linear function over a compact convex set, the minimizer is unique (cf. [36, Theorem 2.2.4]). If K is compact and $\mathbb{R}[x]_{\mathcal{A}}$ is K -full, the set $\mathcal{E}_d(y, K)$ is compact convex by Proposition 3.4, and (3.3) has a minimizer for all R . If $\mathbb{R}[x]_{\mathcal{A}}$ is not K -full, we typically need to choose R that is positive definite on K , to guarantee (3.3) has a minimizer. Therefore, to get an extreme point of $\mathcal{E}_d(y, K)$, it is enough to solve (3.3) for a generic positive definite R , no matter $\mathbb{R}[x]_{\mathcal{A}}$ is K -full or not.

The cone $\mathcal{R}_d(K)$ is typically quite difficult to describe, and generally we can not solve (3.3) directly. Recently, there is much work on solving moment optimization problems like (3.3) by semidefinite relaxations. We refer to [18, 20, 24, 25]. The basic idea is to approximate the cone $\mathcal{R}_d(K)$ by semidefinite programs. So, we use semidefinite relaxations to solve (3.3). This produces a semidefinite algorithm for solving \mathcal{A} -TKMPs.

Suppose K is compact and $K \subseteq B(0, \rho)$. Denote $b := \rho^2 - \|x\|_2^2$. Recall that $h = (h_1, \dots, h_{m_1})$ and $g = (g_1, \dots, g_{m_2})$ are the tuples of polynomials describing K

as in (1.1). For convenience, denote

$$g_B := (g_1, \dots, g_{m_2}, b).$$

The set K can be equivalently described as $h(x) = 0, g_B(x) \geq 0$. Recall the definitions of $Q_k(g)$ in (2.6) and its dual cone $\Phi_k(g)$ in (2.9). Note that

$$Q_k(g_B) = Q_k(g) + Q_k(b), \quad \Phi_k(g_B) = \Phi_k(g) \cap \Phi_k(b).$$

Let $k \geq d/2$ be an integer. The k -th order semidefinite relaxation of (3.3) is

$$(4.1) \quad \min_w \langle R, w \rangle \quad s.t. \quad w|_{\mathcal{A}} = y, w \in \Phi_k(g_B) \cap E_k(h).$$

The dual optimization problem of (4.1) is

$$(4.2) \quad \max_{p \in \mathbb{R}[x]_{2k}} \langle p, y \rangle \quad s.t. \quad R - p \in I_{2k}(h) + Q_k(g_B).$$

For every w feasible for (4.1) and every p feasible for (4.2), we have

$$\langle R, w \rangle \geq \langle p, y \rangle,$$

by weak duality. Thus, the optimal value of (4.1) is always greater than or equal to that of (4.2).

Proposition 4.1. *Let $y \in \mathbb{R}^{\mathcal{A}}$, $d > \deg(\mathcal{A})$, and $K \subseteq B(0, \rho)$ be as in (1.1).*

- (i) *If w^* is a minimizer of (4.1) and has a flat truncation $w^*|_{2t}$ with $2t \geq \deg(\mathcal{A})$, then y admits a finitely atomic K -measure μ .*
- (ii) *If (4.1) is infeasible for some k , then y admits no K -measures.*

Proof. (i) Let $z = w^*|_{2t}$. Then z is a flat tms. So, z admits a finitely atomic K -measure μ , by Theorem 2.2. Since $2t \geq \deg(\mathcal{A})$, $z|_{\mathcal{A}} = y$ and z is an extension of y . Thus, y also admits the measure μ .

(ii) Suppose otherwise y admits a K -measure. By Proposition 3.3, y can be extended to a flat tms $w_1 \in \mathbb{R}^{\mathbb{N}_{2k_1}^{\mathcal{A}}}$ satisfying (2.2). By Theorem 2.2, the tms w_1 admits a r -atomic K -measure, say, $w_1 = c_1[u_1]_{2k_1} + \dots + c_r[u_r]_{2k_1}$ with all $c_i > 0$ and $u_i \in K$. Let $w = c_1[u_1]_{2k} + \dots + c_r[u_r]_{2k}$, then w is feasible for (4.1), a contradiction. So, y admits no K -measures. \square

Proposition 4.1 can be applied to determine whether an \mathcal{A} -tms y admits a K -measure or not. We can start with an order $k \geq d/2$. If (4.1) is infeasible, then we know y admits no K -measures, by Proposition 4.1 (ii). If (4.1) is feasible, we solve it for a minimizer w^* if it exists. If w^* has a flat truncation $w^*|_{2t}$ with $2t \geq \deg(\mathcal{A})$, then y and $w^*|_{2t}$ commonly admit a finitely atomic K -measure. If such a flat truncation does not exist, we increase the order k by one, and repeat the above. This produces the following algorithm for \mathcal{A} -TKMPs.

Algorithm 4.2. A semidefinite algorithm for \mathcal{A} -TKMPs.

Input: An \mathcal{A} -tms y , an even degree $d > \deg(\mathcal{A})$, a semialgebraic set K as in (1.1) and $\rho > 0$ with $K \subseteq B(0, \rho)$.

Output: A finitely atomic K -representing measure for y , or an answer that it does not exist.

Procedure:

Step 0: Choose a generic $R \in \Sigma_{n,d}$ and let $k := d/2$.

Step 1: Solve (4.1). If (4.1) is infeasible, output the answer that y admits no K -measures, and stop. If (4.1) is feasible, get a minimizer $w^{*,k}$. Let $t := \min\{d_K, \deg(\mathcal{A})\}$.

Step 2: Let $z := w^{*,k}|_{2t}$. Check whether the rank condition (2.4) is satisfied or not. If yes, go to Step 3; otherwise, go to Step 4.

Step 3: Compute the finitely atomic measure μ admitted by z :

$$\mu = c_1\delta(u_1) + \cdots + c_r\delta(u_r),$$

where $r = \text{rank} M_t(z)$, each $u_i \in K$ and $c_i > 0$. Output μ , and stop.

Step 4: If $t < k$, set $t := t + 1$ and go to Step 2; otherwise, set $k := k + 1$ and go to Step 1.

For the input, we typically choose $d = 2\lceil(\deg(\mathcal{A}) + 1)/2\rceil$, which is the minimum as required. In Step 0, the genericity means that R is chosen in $\Sigma_{n,d} \setminus \Theta$, for a set $\Theta \subseteq \mathbb{R}^{\mathbb{N}_d^n}$ having zero Lebsgue measure. In implementations, we choose $R = [x]_{d/2}^T G^T G [x]_{d/2}$ with G a random square matrix obeying Gaussian distribution. In Step 2, the rank condition (2.4) is usually checked by using numerical ranks, due to computer round-off errors. In the numerical experiments of this paper, we evaluate the rank of a matrix as the number of its singular values that are greater than or equal to 10^{-6} . In Step 3, the method in [21] can be used to get a r -atomic K -measure for z . Algorithm 4.2 can be easily implemented by using the syntax in software `GloptiPoly 3` [22] and `SeDuMi` [37]. Example 4.4 shows how to do this.

Remark 4.3. If y admits a K -measure, Algorithm 4.2 can produce a r -atomic K -representing measure with $r \leq |\mathcal{A}|$, for *almost all* $R \in \Sigma_{n,d}$, either asymptotically or in finitely many steps (cf. Section 5). The obtained r may not be minimum. It is typically quite difficult to find a representing measure whose support is minimum. This is an important future work, and we do not focus on it here. However, this problem can be treated in some way. For instance, Algorithm 4.2 can be applied repeatedly for a certain number of times, say, N . In each time, a different R is generated, and we typically get different r -atomic measures. Among these N times, we choose the measure whose support is minimum. Our numerical experiments show that this often produces a r -atomic measure with r equal or close to the minimum. Of course, this is heuristic, and there is no theoretical guaranty.

Example 4.4. Consider the \mathcal{A} -tms y with (α, y_α) given as:

α	$\begin{pmatrix} 2 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 2 \end{pmatrix}$	$\begin{pmatrix} 2 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$	$\begin{pmatrix} 2 \\ 2 \end{pmatrix}$	$\begin{pmatrix} 4 \\ 2 \end{pmatrix}$	$\begin{pmatrix} 2 \\ 4 \end{pmatrix}$
y_α	1/3	1/3	0	0	1/9	1/15	1/15

This y admits a measure supported on the square $[-1, 1]^2$, since each y_α is the average of x^α on $[-1, 1]^2$. Clearly, $h = \emptyset$, $g = (1 - x_1^2, 1 - x_2^2)$ and $[-1, 1]^2 \subseteq B(0, \sqrt{2})$. In (4.1), the tuple g_B can be replaced by g , because $\Psi_k(g_B) = \Psi_k(g)$. We apply Algorithm 4.2 to this \mathcal{A} -TKMP. The order $k = 4$ is typically enough to get a flat extension. This can be done by the syntax in `GloptiPoly 3` [22] and `SeDuMi` [37] as follows:

```
mpol x 2;
Amon = [x(1)^2 x(2)^2 x(1)^2*x(2) x(1)*x(2)^2 ...
x(1)^2*x(2)^2 x(1)^4*x(2)^2 x(1)^2*x(2)^4];
y=[ 1/3 1/3 0 0 1/9 1/15 1/15];
conmom = [mom(Amon)==y];
K = [1-x(1)^2>=0, 1-x(2)^2>=0];
bracx = mmon(x,0,4); G = randn(length(bracx));
```

```

R = brax*(G'*G)*brax; k = 4;
P = msdp(min(mom(R)),K,conmom,k);
[A,b,c,S] = msedumi(P);
[xsol,ysol,info] = sedumi(A,b,c,S);
dvar = c-A'*ysol;
Mw = mat( dvar(S.f+1:S.f+S.s(1)^2) );

```

In the above, \mathbf{Mw} is the moment matrix $M_k(w)$. By Remark 4.3, we run Algorithm 4.2 for a couple of times. In each time, the computed \mathbf{Mw} satisfies the rank condition (2.4), and we got a r -atomic measure with $r \leq |\mathcal{A}| = 7$, by the method in [21]. The smallest r we got is 3, which occurred in the representing measure $\sum_{i=1}^3 c_i \delta(u_i)$ with u_i and c_i given as ¹

u_i	(-0.8524, 0.8910)	(0.2109, 0.8873)	(0.6697, -0.3743)
c_i	0.1231	0.2061	0.5233

Example 4.5. Consider the \mathcal{A} -tms y with (α, y_α) given as:

α	$\begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$	$\begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix}$	$\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix}$
y_α	1/5	1/15	1/15	3/105	1/105	1/7

The above y admits a measure on \mathbb{S}^2 , because each y_α is the average of x^α on \mathbb{S}^2 . The sphere \mathbb{S}^2 is defined by $h = (\|x\|_2^2 - 1)$ and $g = \emptyset$. Clearly, $\mathbb{S}^2 \subseteq B(0, 1)$. In (4.1), the tuple $g_B = (1 - \|x\|_2^2)$ can be replaced by $g = \emptyset$, because $\Psi_k(g_B) \cap E_k(h) = \Psi_k(g) \cap E_k(h)$. Like in Example 4.4, we run Algorithm 4.2 for a couple of times. In each time, we got a r -atomic measure with $r \leq |\mathcal{A}| = 6$. The smallest r we got is 2, occurring in the representing measure $\sum_{i=1}^2 c_i \delta(u_i)$ with u_i, c_i given as

u_i	(0.3434 0.4542 0.8221)	(0.8999 0.2550 -0.3539)
c_i	0.4610	0.2952

Example 4.6. (random instances) We apply Algorithm 4.2 to solve some randomly generated \mathcal{A} -TKMPs. Let $K = B(0, 1)$ be the unit ball in \mathbb{R}^n . For each pair (n, m) from $\{(2, 10), (3, 8), (4, 6), (5, 4)\}$, we randomly generate a subset $\mathcal{A} \subseteq \mathbb{N}_m^n$ with cardinality 10, 20, 30 respectively. For each \mathcal{A} , generate $N := \binom{n+m}{m}$ points randomly from the ball $B(0, 1)$, say, u_1, \dots, u_N , and let $y = c_1[u_1]_{\mathcal{A}} + \dots + c_N[u_N]_{\mathcal{A}}$, with $c_i > 0$ random. This is because if a tms in $\mathbb{R}^{\mathbb{N}_m^n}$ admits a measure, then it must admit an N -atomic measure (cf. [1]).

(n, m)	$ \mathcal{A} = 10$	$ \mathcal{A} = 20$	$ \mathcal{A} = 30$
(2, 10)	4,5,6,7,8	8,9,10,11,12,13	9,10,11,12,13,14,15
(3, 8)	4,5,6,7,8,9	9,10,11,12,13,14,15	11,12,13,14,15,16,17,18
(4, 6)	5,6,7,8,9	8,9,10,11,12,13,14	10,11,12,13,14,15,16,17
(5, 4)	3,4,5,6,7,8	7,8,9,10,11,12	10,11,12,13,14,15,16

For each triple $(n, m, |\mathcal{A}|)$, we generate 100 random instances. For every generated instance, Algorithm 4.2 returned a r -atomic representing measure. The values of obtained r are listed in the above table. They are all smaller than $|\mathcal{A}|$. This is justified by Proposition 5.2(ii).

¹Throughout the paper, only four decimal digits are shown for supports and weights.

5. CONVERGENCE ANALYSIS

In this section, we analyze the convergence of Algorithm 4.2. Two kinds of convergence will be investigated: *asymptotic* convergence, and *finite* convergence. For asymptotic convergence, we mean that there exists t such that the truncated sequence $\{w^{*,k}|_{2t}\}$ ($w^{*,k}$ is a minimizer of (4.1) with order k) is bounded and all its accumulation points are flat extensions of y , if y admits a K -measure. For finite convergence, we mean that there exists k such that, either (4.1) is infeasible, or there exists t such that the truncation $w^{*,k}|_{2t}$ is flat. We begin with some properties of the semidefinite relaxations (4.1)-(4.2).

5.1. Properties of the semidefinite relaxations.

Proposition 5.1. *Let $d > \deg(\mathcal{A})$ be even, and $K \subseteq B(0, \rho)$ be as in (1.1). Suppose $y \in \mathbb{R}^{\mathcal{A}}$ admits a K -measure.*

- (i) *If R lies in the interior of $\Sigma_{n,d}$, then, for all $k \geq d/2$, (4.1) is feasible and has a minimizer, and (4.1)-(4.2) have the same optimal value.*
- (ii) *Let R_{\min} be the optimal value of (3.3). Then, the optimal values of (4.1) and (4.2) are smaller than or equal to R_{\min} , for all $k \geq d/2$.*
- (iii) *Suppose R lies in the interior of $\Sigma_{n,d}$. Then, there exists a constant $C = C(R)$ such that, for all w that is a minimizer of (4.1) with order k ,*

$$(5.1) \quad \|w|_{2t}\|_2 \leq (1 + \rho^2 + \dots + \rho^{2t})C, \quad t = 0, 1, \dots, k.$$

Proof. (i) Let μ be a finitely atomic K -representing measure for y , which must exist by Proposition 3.3. Then the tms $\int_K [x]_{2k} d\mu$ is feasible for (4.1). So, (4.1) is feasible for all $k \geq d/2$. Since $R \in \text{int}(\Sigma_{n,d})$, for all $p \in \mathbb{R}[x]_{\mathcal{A}}$ with tiny coefficients, $R - p \in \Sigma_{n,d} \subseteq I_{2k}(h) + Q_k(g_B)$. This means that the zero polynomial 0 is an interior point of (4.2). Hence, strong duality holds, i.e., (4.1) has a minimizer, and (4.1)-(4.2) has the same optimum value, for all $k \geq d/2$ (cf. [2, Section 2.4]).

(ii) Since y admits a K -measure, the feasible set of (3.3) is nonempty. Let z be an arbitrary feasible point of (3.3). Then z admits a finitely atomic K -measure μ , i.e., $z = \int_K [x]_d d\mu$. Let $w = \int_K [x]_{2k} d\mu$. Clearly, $w|_{\mathcal{A}} = y$ and is feasible for (4.1), and $\langle R, z \rangle = \langle R, w \rangle$. That is, (4.1) is a relaxation of (3.3). So, the optimal value of (4.1) is at most R_{\min} . This is also true for the optimal value of (4.2), since it is not bigger than that of (4.1).

(iii) Since $R \in \text{int}(\Sigma_{n,d})$, $R - \epsilon \in \Sigma_{n,d}$ for some $\epsilon > 0$. So, we have

$$0 \leq \langle R - \epsilon, w \rangle = \langle R, w \rangle - \epsilon \langle 1, w \rangle.$$

(Cf. [31, Lemma 2.5].) Since w is a minimizer of (4.1), item (ii) implies that

$$w_0 = \langle 1, w \rangle \leq \langle R, w \rangle / \epsilon \leq R_{\min} / \epsilon.$$

The membership $w \in \Phi_k(g_B)$ implies $L_b^{(k)}(w) \succeq 0$ ($b = \rho^2 - \|x\|_2^2$). So,

$$\rho^2 \mathcal{L}_w(\|x\|_2^{2t-2}) - \mathcal{L}_w(\|x\|_2^{2t}) \geq 0, \quad t = 0, 1, \dots, k.$$

(Cf. [31, Lemma 2.5].) A repeated application of the above gives

$$\mathcal{L}_w(\|x\|_2^{2t}) \leq \rho^{2t} w_0, \quad t = 0, 1, \dots, k.$$

Since $M_k(w) \succeq 0$, for each $t = 0, 1, \dots, k$,

$$\|w|_{2t}\|_2 \leq \|M_t(w)\|_F \leq \text{Trace}(M_t(w)) = \sum_{i=0}^t \sum_{|\alpha|=i} w_{2\alpha},$$

$$\sum_{|\alpha|=i} w_{2\alpha} = \mathcal{L}_w\left(\sum_{|\alpha|=i} x^{2\alpha}\right) \leq \mathcal{L}_w(\|x\|_2^{2i}) \leq \rho^{2i} w_0.$$

Let $C = R_{\min}/\epsilon$, then the inequality (5.1) holds. \square

Proposition 5.2. *Let $K \subseteq \mathbb{R}^n$ be compact and $d > \deg(\mathcal{A})$ be even. Suppose $y \in \mathbb{R}^{\mathcal{A}}$ admits a K -measure. If R is generic in $\Sigma_{n,d}$, then we have:*

- (i) *The problem (3.3) has a unique minimizer.*
- (ii) *If, for some k , a minimizer $w^{*,k}$ of (4.1) has a flat truncation $w^{*,k}|_{2t}$ ($2t \geq d$), then the measure admitted by $w^{*,k}|_{2t}$ is r -atomic with $r \leq |\mathcal{A}|$.*

Proof. The boundary of $\Sigma_{n,d}$ has zero Lebsgue measure in the space $\mathbb{R}^{\mathbb{N}_d^n}$. It is enough to prove the items (i) and (ii) if R is generic in the interior of $\Sigma_{n,d}$. Let

$$S_\ell := \{R \in \Sigma_{n,d} : R - 1/\ell \in \Sigma_{n,d}, \|R\|_2 \leq \ell\}, \quad \ell = 1, 2, \dots$$

Clearly, $\text{int}(\Sigma_{n,d}) = \bigcup_{\ell \geq 1} \text{int}(S_\ell)$. It is sufficient to prove that for each $\ell = 1, 2, 3, \dots$, if R is generic in S_ℓ then the items (i) and (ii) are true.

(i) For every $R \in S_\ell$, we have $1/\ell \leq R(x) \leq \ell\|x\|_d$ for all $x \in \mathbb{R}^n$. Choose a finitely atomic measure $\nu^* \in \text{meas}(y, K)$. Let $M_1 := \int_K \ell\|x\|_d \, d\nu^*$, a constant independent of R . Clearly, $\int_K [x]_d \, d\nu^*$ is feasible in (3.3), and

$$R_{\min} \leq \int_K R \, d\nu^* \leq M_1.$$

For all $R \in S_\ell$, (3.3) has a minimizer z^* (cf. Proposition 3.6), and z^* satisfies

$$z_0^*/\ell \leq \langle R, z^* \rangle = R_{\min} \leq M_1.$$

Note that z^* is feasible for (4.1) with order $k = d/2$. As in the proof of Proposition 5.1(iii), we can get

$$\|z^*\|_2 \leq M_2 := (1 + \rho^2 + \dots + \rho^d)\ell M_1.$$

Thus, for all $R \in S_\ell$, (3.3) is equivalent to

$$(5.2) \quad \min_z \langle R, z \rangle \quad \text{s.t.} \quad z|_{\mathcal{A}} = y, \|z\|_2 \leq M_2, z \in \mathcal{R}_d(K).$$

The feasible set of (5.2), denoted by F , is a nonempty compact convex set. A linear functional $\langle R, z \rangle$ has a unique minimizer on F if and only if R is a regular normal vector of F , or equivalently, $\langle R, z \rangle$ has more than one minimizer on F if and only if R is a singular normal vector of F (cf. Schneider [36, Section 2.2]). Let Θ be the set of all singular normal vectors of F . Then Θ has zero Lebsgue measure in the space $\mathbb{R}^{\mathbb{N}_d^n}$ (cf. Schneider [36, Theorem 2.2.4]). If $R \in S_\ell \setminus \Theta$, then (5.2), as well as (3.3), has a unique minimizer. So, if R is generic in S_ℓ , then (3.3) has a unique minimizer.

(ii) Since $w^{*,k}|_{2t}$ is flat, it admits a finitely atomic K -measure, and so does $w^{*,k}|_d$ when $2t \geq d$. Thus, $w^{*,k}|_d$ is feasible in (3.3), and $\langle R, w^{*,k}|_d \rangle \geq R_{\min}$. By Proposition 5.1(ii), we know $\langle R, w^{*,k}|_d \rangle = \langle R, w^{*,k} \rangle \leq R_{\min}$. Hence, $\langle R, w^{*,k}|_d \rangle = R_{\min}$ and $w^*|_d$ is a minimizer of (3.3). By item (i), we know, if R is generic in $\Sigma_{n,d}$ then (3.3) has a unique minimizer. Therefore, for a generic $R \in \Sigma_{n,d}$, $w^*|_d$ is the unique minimizer of (3.3) and is an extreme point of $\mathcal{E}_d(y, K)$. By Lemma 3.5, every measure admitted by $w^{*,k}|_d$ must be r -atomic with $r \leq |\mathcal{A}|$. Clearly, every measure admitted by $w^{*,k}|_{2t}$ is also admitted by $w^{*,k}|_d$, and thus must be r -atomic with $r \leq |\mathcal{A}|$. \square

5.2. Asymptotic convergence. Our main result in this subsection is:

Theorem 5.3. *Let $d > \deg(\mathcal{A})$ be even and $K \subseteq B(0, \rho)$ be as in (1.1). Suppose $y \in \mathbb{R}^{\mathcal{A}}$ admits a K -measure. In Algorithm 4.2, if R is generic in $\Sigma_{n,d}$, then (4.1) has an optimizer $w^{*,k}$ for every $k \geq d/2$ and we have:*

- (i) *For all t big enough, the sequence $\{w^{*,k}|_{2t}\}$ is bounded and all its accumulation points are flat. Moreover, each of the accumulation points admits a r -atomic measure with $r \leq |\mathcal{A}|$.*
- (ii) *In item (i), if, in addition, d is also big enough, then the sequence $\{w^{*,k}|_{2t}\}$ converges to a flat tms.*

Proof. Since R is generic in $\Sigma_{n,d}$, we can assume $R \in \text{int}(\Sigma_{n,d})$. By Proposition 5.1(i), (4.1) has an optimizer $w^{*,k}$ for every $k \geq d/2$.

- (i) By Proposition 5.1(iii), there is a constant $C = C(R)$ such that

$$\|w^{*,k}|_{2t}\|_2 \leq (1 + \rho^2 + \dots + \rho^{2t})C \quad \text{for all } t = 0, 1, \dots, k.$$

So, the sequence $\{w^{*,k}|_{2t}\}$ is bounded, for any given t . Generally, we can assume $\rho < 1$ (otherwise, scale K to get $\rho < 1$). Let ω be an accumulation point of $\{w^{*,k}|_{2t}\}$. We can generally further assume $w^{*,k}|_{2t} \rightarrow \omega$ as $k \rightarrow \infty$.

First, we prove that the truncation $\omega|_d$ is a minimizer of (3.3). Each $w^{*,k}$ could be thought of as a vector in $\mathbb{R}^{\mathbb{N}_\infty^n}$ by adding zero entries to the tailing. The set $\mathbb{R}^{\mathbb{N}_\infty^n}$ is a Hilbert space, equipped with the inner product

$$\langle u, v \rangle := \sum_{\alpha} u_{\alpha} v_{\alpha}, \quad \forall u, v \in \mathbb{R}^{\mathbb{N}_\infty^n}.$$

In the above, we know that for all k

$$\|w^{*,k}\|_2 = \|w^{*,k}|_{2k}\|_2 \leq (1 + \rho^2 + \dots + \rho^{2k})C \leq C/(1 - \rho).$$

The sequence $\{w^{*,k}\}$ is bounded in $\mathbb{R}^{\mathbb{N}_\infty^n}$. By Alaoglu's Theorem (cf. [7, Theorem V.3.1] or [25, Theorem C.18]), it has a subsequence $\{w^{*,k_j}\}$ that is convergent in the weak-* topology. That is, there exists $w^* \in \mathbb{R}^{\mathbb{N}_\infty^n}$ such that

$$\langle c, w^{*,k_j} \rangle \rightarrow \langle c, w^* \rangle \quad \text{as } j \rightarrow \infty$$

for all $c \in \mathbb{R}^{\mathbb{N}_\infty^n}$. If we choose c as $\langle c, w \rangle = w_{\alpha}$, for each α , then

$$(5.3) \quad w^{*,k_j}|_{\alpha} \rightarrow w^*|_{\alpha}.$$

Since $w^{*,k}|_{2t} \rightarrow \omega$, the above implies $w^*|_{2t} = \omega$. Note that $w^{*,k_j} \in \Phi_{k_j}(g_B) \cap E_{k_j}(h)$ for all k_j . For each $r = 1, 2, \dots$, if $k_j \geq 2r$, then

$$L_{h_i}^{(r)}(w^{(k_j)}) = 0 \quad (1 \leq i \leq m_1), \quad L_{g_i}^{(r)}(w^{(k_j)}) \succeq 0 \quad (0 \leq i \leq m_2).$$

Hence, (5.3) implies that for all $r = 1, 2, \dots$

$$L_{h_i}^{(r)}(w^*) = 0 \quad (1 \leq i \leq m_1), \quad L_{g_i}^{(r)}(w^*) \succeq 0 \quad (0 \leq i \leq m_2).$$

This means that $w^* \in \mathbb{R}^{\mathbb{N}_\infty^n}$ is a full moment sequence whose localizing matrices of all orders are positive semidefinite. By Lemma 3.2 of Putinar [29], w^* admits a K -measure. So, the truncation $\omega|_d = w^*|_d$ is feasible for (3.3), and

$$R_{\min} \leq \langle R, w^*|_d \rangle = \langle R, w^* \rangle = \langle R, \omega|_d \rangle.$$

By Proposition 5.1(ii), we know $\langle R, w^{*,k} \rangle \leq R_{\min}$ for all k . Thus,

$$\langle R, w^* \rangle = \lim_{j \rightarrow \infty} \langle R, w^{*,k_j} \rangle \leq R_{\min}.$$

Hence, $\langle R, \omega|_d \rangle = R_{min}$, and $\omega|_d$ is a minimizer of (3.3).

Second, we prove that if $t \geq |\mathcal{A}|d_K$ then the truncation $\omega|_{2t}$ is flat. By Proposition 5.2, if R is generic in $\Sigma_{n,d}$, then (3.3) has a unique minimizer, which must be $\omega|_d$, by the above. So, $\omega|_d$ is an extreme point of $\mathcal{E}_d(y, K)$ which is the feasible set of (3.3). Let μ^* be a K -representing measure for ω , which must exist because $\omega = w^*|_{2t}$. Then $\mu^* \in meas(\omega|_d, K)$. By Lemma 3.5, μ^* must be finitely atomic and $|\text{supp}(\mu^*)| \leq |\mathcal{A}|$. Note that

$$\begin{aligned} \text{rank } M_0(\omega) &\leq \text{rank } M_{d_K}(\omega) \leq \cdots \leq \text{rank } M_{d_K|\mathcal{A}|}(\omega) \leq \cdots, \\ \text{rank } M_i(\omega) &\leq |\text{supp}(\mu^*)| \leq |\mathcal{A}|, \quad i = 1, 2, \dots \end{aligned}$$

There must exist $\ell \leq |\mathcal{A}|$ such that

$$\text{rank } M_{(\ell-1)d_K}(\omega) = \text{rank } M_{\ell d_K}(\omega).$$

So, the truncation $\omega|_{2\ell d_K}$ is flat. By Theorem 2.2, it admits a unique K -representing measure. Note that μ^* is a K -representing measure for $\omega|_{2\ell d_K}$, and it is unique. Clearly, ω is an extension of $\omega|_{2\ell d_K}$, and every measure admitted by ω is also a representing measure for $\omega|_{2\ell d_K}$. Therefore, μ^* is also the unique K -representing measure for ω . Clearly, μ^* is r -atomic and $r = \text{rank } M_{\ell d_K}(\omega)$. So,

$$\begin{aligned} |\text{supp}(\mu^*)| &= \text{rank } M_{\ell d_K}(\omega) \leq \\ \text{rank } M_{\ell d_K+1}(\omega) &\leq \cdots \leq \text{rank } M_t(\omega) \leq |\text{supp}(\mu^*)|, \end{aligned}$$

and we have $\text{rank } M_{t-d_K}(\omega) = \text{rank } M_t(\omega)$, i.e., ω is flat.

Third, in the above, we have indeed shown that if μ^* is a K -representing measure for any accumulation point ω , then μ^* is r -atomic with $r \leq |\mathcal{A}|$.

(ii) It is enough to show that if $t \geq d/2 \geq |\mathcal{A}|d_K$ then $\{w^{*,k}|_{2t}\}$ has a unique accumulation point. We continue the proof of the item (i). By Proposition 5.2, if R is generic in $\Sigma_{n,d}$, then (3.3) has a unique minimizer, say, z^* . Let ω be an arbitrary accumulation point of $\{w^{*,k}|_{2t}\}$. We have shown that $\omega|_d$ is a minimizer of (3.3) in (i). So, $\omega|_d = z^*$. If $d \geq 2|\mathcal{A}|d_K$, we have shown that z^* is flat in (i). By Theorem 2.2, z^* admits a unique K -representing measure, say, ν^* . Since $\omega|_d = z^*$, every K -measure admitted by ω must also be a K -representing measure for z^* . Hence, ν^* must be the unique K -representing measure for ω , i.e., $\omega = \int_K [x]_{2t} d\nu^*$. This shows that $\{w^{*,k}|_{2t}\}$ has a unique accumulation point, which is $\int_K [x]_{2t} d\nu^*$. \square

Algorithm 4.2 is guaranteed to converge with *probability one*, for all \mathcal{A} -tms y that admits a K -measure. If, accidentally, a bad R is generated such that Algorithm 4.2 fails to converge, we can choose a different generic $R \in \Sigma_{n,d}$. Indeed, this never happened in our numerical experiments. Theorem 5.3 guarantees that we *almost always* succeed by choosing R from $\Sigma_{n,d}$.

5.3. Finite convergence. Now we characterize when Algorithm 4.2 has finite convergence. Denote $g_{m_2+1} := \rho - \|x\|_2^2$, then $g_B = (g_1, \dots, g_{m_2}, g_{m_2+1})$. For an index set $J = \{j_1, \dots, j_r\}$, denote $g_J := (g_{j_1}, \dots, g_{j_r})$.

For a polynomial f , denote by f^{hom} the homogeneous part of f with the highest degree. If $p = (p_1, \dots, p_r)$ is tuple, denote $p^{hom} := (p_1^{hom}, \dots, p_r^{hom})$. We denote by $Jac(p)|_u$ the Jacobian of p evaluated at the point u . The discriminant of p^{hom} , denoted as $\Delta(p^{hom})$, is a polynomial in the coefficients of p^{hom} , such that $\Delta(p^{hom}) = 0$ if and only if $p^{hom}(x) = 0$ has a nonzero solution $u \in \mathbb{C}^n$ with $\text{rank } Jac(p^{hom})|_u < r$. We refer to [30, Section 3] for discriminants.

Assumption 5.4. For any $J \subseteq [m_2 + 1]$ with $V_{\mathbb{R}}(h, g_J) \neq \emptyset$, the coefficients of h and g_B satisfy $\Delta(h^{hom}, g_J^{hom}) \neq 0$.

Assumption 5.4 holds generically, because it requires (h^{hom}, g^{hom}) to lie in an open dense set. The finite convergence of Algorithm 4.2 is as follows.

Theorem 5.5. Let $K \subseteq B(0, \rho)$ be as in (1.1), and y be an \mathcal{A} -tms. Let $w^{*,k}$ be an optimizer of (4.1) with order k , if it exists.

- (i) Suppose $\mathbb{R}[x]_{\mathcal{A}}$ is K -full. If y admits no K -measures, then (4.1) is infeasible for all k big enough.
- (ii) Suppose $\text{meas}(y, K) \neq \emptyset$ and Assumption 5.4 holds. If (3.4) has a maximizer p^* with $R - p^* \in I_{2k_1}(h) + Q_{k_1}(g_B)$ for some k_1 , and if R is generic in $\Sigma_{n,d}$, then $w^{*,k}|_{2t}$ is flat for some $t \geq d/2$, for k big enough.
- (iii) Suppose (4.2) achieves its maximum for k big enough. If the truncation $w^{*,k_2}|_{2t}$ is flat for some k_2 and $t \geq d/2$, and if $R \in \text{int}(\Sigma_{n,d})$, then (3.4) has a maximizer p^* with $R - p^* \in I_{2k_3}(h) + Q_{k_3}(g_B)$ for some k_3 .

Proof. (i) If $\mathbb{R}[x]_{\mathcal{A}}$ is K -full and $y \notin \mathcal{R}_{\mathcal{A}}(K)$, by Corollary 3.2, there exists $p_1 \in \mathbb{R}[x]_{\mathcal{A}}$ such that $p_1|_K \geq 0$, $\langle p_1, y \rangle < 0$. By the K -fullness of $\mathbb{R}[x]_{\mathcal{A}}$, there exists $p_2 \in \mathbb{R}[x]_{\mathcal{A}}$ with $p_2|_K > 0$. So, for $\epsilon > 0$ small, $\hat{p} := p_1 + \epsilon p_2$ satisfies $\hat{p}|_K > 0$, $\langle \hat{p}, y \rangle < 0$. Let $\eta_0 > 0$ be such that $(R + \eta_0 \hat{p})|_K > 0$. By Theorem 2.4, both $R + \eta_0 \hat{p}$ and \hat{p} belong to $I_{2t_1}(h) + Q_{t_1}(g_B)$, for some t_1 . Hence, for all $\eta > \eta_0$ and $k \geq t_1$, $-\eta \hat{p}$ is feasible for (4.2), because

$$R + \eta \hat{p} = R + \eta_0 \hat{p} + (\eta - \eta_0) \hat{p} \in I_{2k}(h) + Q_k(g_B).$$

Note that $\langle -\eta \hat{p}, y \rangle \rightarrow +\infty$ as $\eta \rightarrow +\infty$. So, the optimal value of (4.2) is $+\infty$. By weak duality, its dual problem (4.1) must be infeasible for all $k \geq t_1$.

(ii) By Proposition 5.1(i), (4.1) and (4.2) have the same optimal value, for all k , if $R \in \text{int}(\Sigma_{n,d})$ (this is true if R is generic in $\Sigma_{n,d}$). For all $k \geq k_1$, p^* is feasible for (4.2). So, if $k \geq k_1$, then $\langle R, w^{*,k} \rangle = \langle p^*, y \rangle = \langle p^*, w^{*,k} \rangle$, and $\langle R - p^*, w^{*,k} \rangle = 0$. Let $q := R - p^*$. Then

$$\langle q, w^{*,k} \rangle = 0 \quad \forall k \geq k_1.$$

Clearly, q is nonnegative on K . Let z^* be a minimizer of (3.3) and $\mu \in \text{meas}(z, K)$. Then, by Proposition 3.6 (i) (note $R|_K > 0$ if R is generic in $\Sigma_{n,d}$),

$$0 = \langle R, z^* \rangle - \langle p^*, y \rangle = \langle q, z^* \rangle = \int_K q d\mu.$$

Thus, q vanishes on $\text{supp}(\mu)$, and has a zero on K .

Consider the optimization problem:

$$(5.4) \quad \min_x q(x) \quad \text{s.t.} \quad h(x) = 0, g_B(x) \geq 0.$$

The k -th order SOS relaxation for (5.4) is

$$(5.5) \quad \gamma_k := \max \gamma \quad \text{s.t.} \quad q - \gamma \in I_{2k}(h) + Q_k(g_B).$$

Its dual problem is

$$(5.6) \quad \min_w \langle q, w \rangle \quad \text{s.t.} \quad w \in \Phi_k(g_B) \cap E_k(h), w_0 = 1.$$

The minimum of q over K is 0, and $\gamma_k = 0$ for all $k \geq k_1$. Thus, the sequence $\{\gamma_k\}$ has finite convergence. The SOS program (5.5) achieves its optimal value for $k \geq k_1$, because $q \in I_{2k_1}(h) + Q_{k_1}(g_B)$.

Since $d > \deg(\mathcal{A})$, $(R - p^*)^{hom} = R^{hom}$. Under Assumption 5.4, for every $J \subseteq [m_2 + 1]$ with $V_{\mathbb{R}}(h, g_J) \neq \emptyset$, $\Delta(f, h^{hom}, g_J^{hom})$ is not constantly zero in f (cf. [30, Theorem 3.2]). So, if R is generic in $\Sigma_{n,d}$, then $\Delta(R^{hom}, h^{hom}, g_J^{hom}) \neq 0$ for all such J . By Proposition A.1 in the Appendix, (5.4) has only finitely many critical varieties and Assumption 2.1 in [31] for (5.4) is satisfied². If $w_0^{*,k} = 0$, then $w_\alpha^{*,k} = 0$ for all α because $M_k(w^{*,k}) \succeq 0$, and $w^{*,k}$ is clearly flat. If $w_0^{*,k} > 0$, we can scale $w^{*,k}$ so that $w_0^{*,k} = 1$. Then $w^{*,k}$ is a minimizer of (5.6) because $\langle q, w^{*,k} \rangle = 0$ for all $k \geq k_1$. By Theorem 2.2 of [31], $w^{*,k}$ has a flat truncation $w^{*,k}|_{2t}$ if k is big enough. Indeed, $w^{*,k}|_{2k-2}$ is flat (cf. Remark 2.3 of [31]). So, there is a flat truncation $w^{*,k}|_{2t}$ with $t \geq d/2$ if k is big enough.

(iii) Suppose $w^{*,k_2}|_{2t}$ is flat and $2t \geq d$. Let R_{min} be the optimal value of (3.3), then by Proposition 5.1(ii), $R_{min} \geq \langle R, w^{*,k_2} \rangle$. On the other hand, the truncation $w^{*,k_2}|_d$ is feasible in (3.3), so $R_{min} \leq \langle R, w^{*,k_2} \rangle$. Thus, $R_{min} = \langle R, w^{*,k_2} \rangle$. Indeed, we have $R_{min} = \langle R, w^{*,k} \rangle$ for all $k \geq k_2$. By assumption, (4.2) has a maximizer p^* at a big order, say, $k_3 \geq k_2$. Then $R - p^* \in I_{2k_3}(h) + Q_{k_3}(g_B)$. Clearly, p^* is feasible for (3.4) because $(R - p^*)|_K \geq 0$. By Proposition 5.1(i), the optimal value of (4.2) is also equal to R_{min} , if $R \in \text{int}(\Sigma_{n,d})$. So, $\langle p^*, y \rangle = \langle R, w^{*,k_3} \rangle = R_{min}$. The optimal values of (3.3) and (3.4) are equal, by Proposition 3.6(i) if $R \in \text{int}(\Sigma_{n,d})$. Hence, p^* is a maximizer of (3.4). \square

Remark 5.6. By item (i) of Theorem 5.5, if $\mathbb{R}[x]_{\mathcal{A}}$ is K -full and y admits no K -measures, then (4.1) is infeasible for some k , for *any* R (we don't need $R \in \Sigma_{n,d}$). When $\mathbb{R}[x]_{\mathcal{A}}$ is not K -full and y admits no K -measures, it is not clear whether or not there exists k such that (4.1) is infeasible. This is because there does not exist a characterization like Theorem 3.1 for the membership in $\mathcal{R}_{\mathcal{A}}(K)$ if $\mathbb{R}[x]_{\mathcal{A}}$ is not K -full, to the best of the author's knowledge. Theorem 3.1 and Corollary 3.2 assume K -fullness of $\mathbb{R}[x]_{\mathcal{A}}$. In many applications, $\mathbb{R}[x]_{\mathcal{A}}$ is often K -full. On the other hand, if y admits a K -measure, no matter $\mathbb{R}[x]_{\mathcal{A}}$ is K -full or not, for a generic $R \in \Sigma_{n,d}$, Algorithm 4.2 will find a finitely atomic K -representing measure for y , either asymptotically or in finitely many steps.

Remark 5.7. a) When $\text{int}(K) \neq \emptyset$, (4.1) has interior points, and thus (4.2) achieves its optimal value, for every order k (cf. [20, 25]). b) By Theorem 5.5 (ii) and (iii), the condition $R - p^* \in I(h) + Q(g_B)$ is almost necessary and sufficient for finite convergence to occur, modulo some general technical assumptions. c) If a polynomial f is nonnegative on K , then $f \in I(h) + Q(g_B)$, under some general conditions (cf. [32]). So, the condition $R - p^* \in I(h) + Q(g_B)$ is often satisfied. Thus, it is very likely that Algorithm 4.2 has finite convergence. Indeed, the finite convergence occurred in all our numerical experiments.

6. APPLICATIONS

In this section, we show how Algorithm 4.2 can be applied to solve CP/SOEP-decomposition problems and the standard truncated K -moment problems.

²In [31], polynomial optimization problems with only inequality constraints were discussed. If there are equality constraints, Assumption 2.1 in [31] can be naturally modified to include all constraining equations, and Theorem 2.2 of [31] is still true, with the same proof.

6.1. Completely positive matrices. Recall that a matrix $C \in \mathcal{S}^n$ is completely positive if there exist $u_1, \dots, u_r \in \mathbb{R}_+^n$ such that

$$(6.1) \quad C = u_1 u_1^T + \dots + u_r u_r^T.$$

If (6.1) holds, we say C is a CP-matrix. The number r is called the length of the decomposition (6.1). The smallest r , for which (6.1) holds, is called the *CP-rank* of C (cf. [5]). Let $\mathbf{Cp}(n)$ be the cone of $n \times n$ CP-matrices. Clearly, $C \in \mathbf{Cp}(n)$ if and only if $C = BB^T$ for a *nonnegative matrix* B (i.e., every entry of B is nonnegative). So, every CP-matrix must be positive semidefinite, but typically not vice versa. The dual cone of $\mathbf{Cp}(n)$ is the set of $n \times n$ copositive matrices (a matrix $A \in \mathcal{S}^n$ is copositive if $x^T A x \geq 0$ for all $x \in \mathbb{R}_+^n$). Let $\Delta_n = \{x \in \mathbb{R}_+^n : x_1 + \dots + x_n = 1\}$ be the standard simplex in \mathbb{R}^n .

Completely positive and copositive matrices have wide applications in optimization, like approximating stability numbers (cf. [13]) or solving nonconvex quadratic programs (cf. [6]). Checking the membership in $\mathbf{Cp}(n)$ is NP-hard (cf. [14]). We refer to the survey [16] by Dür and the book [5] by Berman and Shaked-Monderer. If C is not a CP-matrix, how can we get a certificate for that? If it is, how can we get a CP-decomposition for C ? When C is acyclic or circular, Dickinson and Dür [15] showed that checking complete positivity can be done in linear-time. In the general case, Berman and Rothblum [4] showed that checking complete positivity and computing CP-ranks can be done by using Renegar's algorithm on quantifier elimination [33]. This method is symbolic, typically runs in exponential time, and is usually very expensive to implement. In the prior existing work, there are no much efficient numerical methods for solving general CP-decomposition problems, in the author's best knowledge.

Clearly, C is a CP-matrix if and only if

$$(6.2) \quad C = \varrho_1 u_1 u_1^T + \dots + \varrho_r u_r u_r^T,$$

for some $u_1, \dots, u_r \in \Delta_n$, $\varrho_1, \dots, \varrho_r > 0$. Every symmetric matrix C can be identified by the vector consisting of its entries

$$\mathbf{c} = (C_{ij})_{i \leq j}.$$

Let $\mathcal{Q}_n = \{\alpha \in \mathbb{N}^n : |\alpha| = 2\}$. Then \mathbf{c} is a \mathcal{Q}_n -tms, and (6.2) is equivalent to

$$\mathbf{c} = \varrho_1 [u_1]_{\mathcal{Q}_n} + \dots + \varrho_r [u_r]_{\mathcal{Q}_n}.$$

Clearly, if $C \in \mathbf{Cp}(n)$, then \mathbf{c} admits a Δ_n -measure. Conversely, if \mathbf{c} admits a Δ_n -measure, then \mathbf{c} also admits a finitely atomic Δ_n -measure like the above (cf. Proposition 3.3), and $C \in \mathbf{Cp}(n)$. Thus, the CP-decomposition problem is essentially an \mathcal{A} -TKMP with $\mathcal{A} = \mathcal{Q}_n$, $K = \Delta_n$. The simplex Δ_n is in the form (1.1), with $h = (\mathbf{1}^T x - 1)$ and $g = (x_1, \dots, x_n)$ ($\mathbf{1}$ denotes the vector of all ones), and $\Delta_n \subseteq B(0, 1)$. Note that $\mathbb{R}[x]_{\mathcal{Q}_n}$ is Δ_n -full.

By the above, the CP-decomposition problem can be solved by Algorithm 4.2. If $C \notin \mathbf{Cp}(n)$, then Algorithm 4.2 will return a certificate for this (i.e., (4.1) is infeasible for some k), by Theorem 5.5(i). If $C \in \mathbf{Cp}(n)$, then we can asymptotically get a flat extension of \mathbf{c} , for almost all $R \in \Sigma_{n,d}$ ($d > 2$ is even), by Theorem 5.3. Moreover, we can likely get it within finitely many steps (cf. Remark 5.7). Indeed, finite convergence occurred in all our numerical experiments. After getting a flat extension of \mathbf{c} , we can get a r -atomic Δ_n -representing measure for \mathbf{c} , which then produces a CP-decomposition for C .

Example 6.1. Consider the matrix:

$$C = \begin{bmatrix} 6 & 4 & 1 & 2 & 2 \\ 4 & 6 & 0 & 1 & 3 \\ 1 & 0 & 3 & 1 & 2 \\ 2 & 1 & 1 & 2 & 1 \\ 2 & 3 & 2 & 1 & 5 \end{bmatrix}.$$

We apply Algorithm 4.2 to the corresponding \mathcal{Q}_5 -tms

$$\mathbf{c} := (6, 4, 1, 2, 2, 6, 0, 1, 3, 3, 1, 2, 2, 1, 5).$$

To get a decomposition of small length, we run Algorithm 4.2 for a couple of times (cf. Remark 4.3). In each time, we got a CP-decomposition for C . The smallest length we got is 5, which occurred in the factorization $C = BB^T$ with

$$B = \begin{bmatrix} 1.0911 & 2.0836 & 0.0000 & 0.3148 & 0.6076 \\ 0.0000 & 1.6456 & 0.0000 & 1.8143 & 0.0000 \\ 0.1488 & 0.0000 & 1.0379 & 0.0000 & 1.3786 \\ 1.0797 & 0.3606 & 0.8087 & 0.2241 & 0.0000 \\ 0.5830 & 0.0000 & 0.0000 & 1.6535 & 1.3878 \end{bmatrix}.$$

The CP-rank of C is 5, because $5 = \text{rank } C \leq \text{CP-rank } C \leq 5$.

Example 6.2. Consider the matrix (cf. [5, Example 2.9]):

$$C = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 1 & 0 & 0 & 1 & 6 \end{bmatrix}.$$

It is positive semidefinite, but not completely positive (cf. [5]). We apply Algorithm 4.2 to verify this fact. It terminates at Step 2 with $k = 2$, because (4.1) is infeasible. This confirms that C is not a CP-matrix.

Example 6.3. (random instances) We apply Algorithm 4.2 to randomly generated CP matrices. If $C \in \mathbf{Cp}(n)$, then C admits a CP-decomposition (6.2) with length $r \leq \frac{1}{2}n(n+1)$, by Carathéodory's Theorem. Indeed, it can be slightly sharpened to $r \leq \frac{1}{2}\text{rank } C(\text{rank } C + 1) - 1$, if $\text{rank } C \geq 2$ (cf. [3, 28]). Clearly, we always have $r \geq \text{rank}(C)$. So, if $C \in \mathbf{Cp}(n)$ and C has full rank, then $n \leq r \leq cp(n) := \frac{1}{2}n(n+1) - 1$, for $n > 1$. For $n = 2, 3, \dots, 8$, we generate 50 instances, except for $n = 8$ (only 20 instances are generated). For each instance, generate $N := \frac{1}{2}n(n+1)$ points randomly from Δ_n , say, u_1, \dots, u_N , and let $C = c_1 u_1 u_1^T + \dots + c_N u_N u_N^T$ with $c_i > 0$ random. For each C , we apply Algorithm 4.2 ten times and let r be the smallest length that is obtained. Algorithm 4.2 is able to get a CP-decomposition for all generated C . The obtained values of r are listed in the table:

n	2	3	4	5	6	7	8
$cp(n)$	2	5	9	14	20	27	35
r	2	3	4	5,6	6,7,8	8,9,10	11,12,13,14,15

They are equal or close to the lower bound n (because $\text{rank } C = n$ for generated C), and is much less than the upper bound $cp(n)$ for $n \geq 4$.

6.2. Sum of even powers (SOEP) of real linear forms. Recall that a form f of an even degree m is SOEP if for some real linear forms L_1, \dots, L_r

$$(6.3) \quad f = L_1^m + \dots + L_r^m$$

Let $Q_{n,m}$ denote the set of all SOEP forms in n variables and of degree m . Reznick proved that $Q_{n,m}$ is a convex cone with nonempty interior and its dual cone is the set of nonnegative forms in n variables and of degree m . We refer to Reznick [35] for SOEP forms. The number of sums, r , is called the length of the SOEP-decomposition (6.3). The minimum r for which (6.3) holds is called the *width* of f , and is denoted as $w(f)$ (cf. [35]). The decomposition (6.3) is called *minimum* if $r = w(f)$. SOEP decompositions naturally have wide and interesting applications, like in Waring's problems, quadrature problems, sphere designs [35]. It is typically quite difficult to check whether a form is SOEP or not. As shown by Reznick [35], when $m \geq 4$, a rational form $f \in Q_{n,m}$ may not have a decomposition (6.3) with all L_i rational. Therefore, numerical methods are preferable in applications. In the prior existing work, there are no much efficient numerical methods for SOEP-decompositions, in the author's best knowledge.

Denote $\mathbb{H}_m^n := \{\alpha \in \mathbb{N}^n : |\alpha| = m\}$. We can write a form f of degree m as

$$f = \sum_{\alpha \in \mathbb{H}_m^n} \binom{m}{\alpha} \check{f}_\alpha x^\alpha.$$

Denote $\check{f} := (\check{f}_\alpha)$. So, f can be identified by the \mathbb{H}_m^n -tms \check{f} . If f is SOEP and (6.3) holds, then we can write each $L_i = \sqrt[m]{c_i}(u_i^T x)$ with $c_i > 0$ and

$$u_i \in \mathbb{S}_+^{n-1} := \{x \in \mathbb{S}^{n-1} : \mathbf{1}^T x \geq 0\}.$$

Thus, we get

$$f = \sum_{i=1}^r \sum_{\alpha \in \mathbb{H}_m^n} \binom{m}{\alpha} c_i u_i^\alpha x^\alpha.$$

The above is equivalent to the decomposition:

$$(6.4) \quad \check{f} = c_1[u_1]_{\mathbb{H}_m^n} + \dots + c_r[u_r]_{\mathbb{H}_m^n}.$$

Clearly, if f is SOEP, then \check{f} admits a \mathbb{S}_+^{n-1} -measure. Conversely, if \check{f} admits an \mathbb{S}_+^{n-1} -measure, then \check{f} also admits a finitely atomic \mathbb{S}_+^{n-1} -measure (cf. Proposition 3.3), and so f is SOEP. Hence, checking $f \in Q_{n,m}$ is equivalent to determining whether the \mathbb{H}_m^n -tms \check{f} admits a \mathbb{S}_+^{n-1} -measure or not. The latter is a \mathcal{A} -TKMP with $\mathcal{A} = \mathbb{H}_m^n$ and $K = \mathbb{S}_+^{n-1}$. Note that $\mathbb{R}[x]_{\mathbb{H}_m^n}$ is \mathbb{S}_+^{n-1} -full.

SOEP-decomposition problems can be solved by Algorithm 4.2. The set \mathbb{S}_+^{n-1} is as in (1.1) with $h = (\|x\|_2^2 - 1)$ and $g = (\mathbf{1}^T x)$. Clearly, $\mathbb{S}_+^{n-1} \subseteq B(0, 1)$. In (4.1), the tuple g_B can be replaced by g , because $\Psi_k(g) \cap E_k(h) = \Psi_k(g_B) \cap E_k(h)$. If f is not SOEP, then \check{f} admits no \mathbb{S}_+^{n-1} -measures, and Algorithm 4.2 can give a certificate for this (i.e., (4.1) is infeasible for some order k), by Theorem 5.5(i). If f is SOEP, then we can asymptotically get a flat extension of \check{f} , for almost all $R \in \Sigma_{n,d}$ ($d > m$ is even), by Theorem 5.3. Moreover, we can likely get it within finitely many steps (cf. Remark 5.7). Indeed, this occurred in all our numerical experiments. Once a flat extension of \check{f} is obtained, we can easily get an SOEP-decomposition for f from a finitely atomic representing measure for \check{f} .

Example 6.4. Consider the sextic form

$$q_\lambda := (x_1^2 + x_2^2 + x_3^2)^3 - \lambda(x_1^6 + x_2^6 + x_3^6).$$

It is SOEP if and only if $\lambda \leq 2/3$ (cf. [35, p. 146]). For $\lambda = 2/3$, by running Algorithm 4.2 a few times, we got an SOEP decomposition of length 10 for $q_{2/3}$:

$$\frac{1}{15} ((x_1 + x_2)^6 + (x_2 + x_3)^6 + (x_1 + x_3)^6 + (x_1 - x_2)^6 + (x_1 - x_3)^6 + (x_2 - x_3)^6) + \frac{1}{60} ((x_1 + x_2 + x_3)^6 + (-x_1 + x_2 + x_3)^6 + (x_1 - x_2 + x_3)^6 + (x_1 + x_2 - x_3)^6).$$

For $\lambda = 1/3$, we can get an SOEP-decomposition of length 11 for $q_{1/3}$, by the same way. The lengths 10 and 11 are the smallest ones that we can get for $q_{2/3}$ and $q_{1/3}$ respectively (cf. Remark 5.7). When $\lambda = 1$, (4.1) is infeasible for $k = 4$, and Algorithm 4.2 terminates at Step 2. This confirms $q_1 \notin Q_{3,6}$.

Example 6.5. (random instances) We apply Algorithm 4.2 to randomly generated SOEP forms. Let m be an even degree. If $f \in Q_{n,m}$, then its width $w(f) \leq N := \binom{n+m-1}{m}$, by Carathéodory's Theorem. If $f \in \text{int}(Q_{n,m})$, then its width $w(f) \geq N_0 := \binom{n+m/2-1}{m/2}$ (cf. [35, Theorem 3.14(iv)]). If $n = 2$ or $(n, m) = (3, 4)$, then $w(f) \leq N_0$ (cf. [35, Theorem 4.6]). So, for the above ranges of (n, m) , we know the generic width is N_0 . For other ranges of (n, m) , if f is generic inside $Q_{n,m}$, then $N_0 \leq w(f) \leq N$. We consider (n, m) from the table:

(n, m)	(2,4)	(2,6)	(2,8)	(2,10)	(3,4)	(4,4)	(3,6)	(3,8)
gwidth	3	4	5	6	6	[10,35]	[10,28]	[15,45]
r	3	4	5	6	6,7	12,13,14	11,12	17,18,19

In the above, **gwidth** is N_0 if $n = 2$ or $(n, m) = (3, 4)$, and is the range $[N_0, N]$ for other cases. For each pair (n, m) from the above table, we generate 50 instances. In each instance, generate points u_1, \dots, u_N randomly from \mathbb{S}^{n-1} , and let $f = c_1(u_1^T x)^m + \dots + c_N(u_N^T x)^m$ with $c_i > 0$ random. For each generated f , we run Algorithm 4.2 for ten times, and choose r to be the smallest length of the obtained SOEP-decompositions. For all generated f , we got an SOEP-decomposition, and the values of obtained lengths r are listed in the above table. We can see that r is equal or close to the minimum.

6.3. Standard truncated K -moment problems. When $\mathcal{A} = \mathbb{N}_m^n$, the \mathcal{A} -TKMP is specialized to the standard truncated K -moment problem (TKMP), which was originally studied by Curto and Fialkow [8, 9, 11, 12]. Algorithm 4.2 can be naturally applied to solve TKMPs. The set $\mathbb{R}[x]_m$ is K -full, for any set K . We are interested in the case that K is compact. If a tms $y \in \mathbb{R}^{\mathbb{N}_m^n}$ admits no K -measures, Algorithm 4.2 can return a certificate for the nonexistence of representing measures; if y admits a K -measure, we can asymptotically get a flat extension of y , for almost all $R \in \Sigma_{n,d}$ ($d > m$ is even), by Theorems 5.3; moreover, we can likely get it within finitely many steps (cf. Theorem 5.5 and Remark 5.7). In the author's best knowledge, Algorithm 4.2 is the first numerical algorithm that can solve general TKMPs with a compact semialgebraic set K .

Example 6.6. Consider the following tms in $\mathbb{R}^{\mathbb{N}_6^2}$:

$$(1, 0, 0, 1/3, 0, 1/3, 0, 0, 0, 0, 1/5, 0, 1/9, 0, 1/5, 0, 0, 0, 0, 0, 1/7, 0, 1/15, 0, 1/15, 0, 1/7).$$

Its moments are listed in the graded lexicographical ordering. This tms admits a measure supported on $[-1, 1]^2$, because its α -th moment is the mean value of x^α on $[-1, 1]^2$. We apply Algorithm 4.2 to this tms. In each time of running, we got

a r -atomic representing measure supported in $[-1, 1]^2$. After a repeated running, the smallest r we got is 10 (cf. Remark 4.3), which occurred in the representing measure $\sum_{i=1}^{10} c_i \delta(u_i)$ with u_i and c_i given as:

u_i	c_i	u_i	c_i
(-0.7983, -0.9666)	0.0318	(-0.8710, -0.2228)	0.0947
(-0.2155, -0.6211)	0.1698	(-0.9175, 0.8643)	0.0328
(0.5833, -0.8876)	0.0729	(-0.4834, 0.4714)	0.1662
(0.3541, 0.0676)	0.2054	(0.9269, -0.3819)	0.0672
(0.0841, 0.9294)	0.0717	(0.8153, 0.6919)	0.0874

Acknowledgement The author was partially supported by the NSF grant DMS-0844775. He would like very much to thank Raman Sanyal and Bernd Sturmfels for comments on generic linear optimization over compact convex sets.

APPENDIX A. GENERIC FINITENESS OF CRITICAL VARIETIES

Consider the polynomial optimization problem

$$(A.1) \quad \begin{cases} \min & f(x) \\ \text{s.t.} & h_i(x) = 0 \ (i \in [m_1]), \ g_j(x) \geq 0 \ (j \in [m_2 + 1]). \end{cases}$$

For each $J \subseteq \{1, \dots, m_2 + 1\}$, denote

$$\mathcal{V}_J := \{x \in \mathbb{C}^n : h(x) = 0, g_J(x) = 0, \text{rank } Jac(f, h, g_J)|_x \leq m_1 + |J|\}.$$

The set of critical points of (A.1) with the active set J is contained in \mathcal{V}_J , which is called a critical variety of (A.1). We show that if the coefficients of $f^{hom}, h^{hom}, g_J^{hom}$ satisfy some discriminantal inequalities, then \mathcal{V}_J is finite. We refer to [30, Section 3] for the definition of discriminants Δ .

Proposition A.1. *Let f, h_i ($i \in [m_1]$), g_j ($j \in [m_2 + 1]$) $\in \mathbb{R}[x]$, and \mathcal{V}_J be defined as above. For any $J \subseteq [m_2 + 1]$, if*

$$\Delta(f^{hom}, h^{hom}, g_J^{hom}) \neq 0, \quad \Delta(h^{hom}, g_J^{hom}) \neq 0,$$

then \mathcal{V}_J is finite.

Proof. Denote $\tilde{x} := (x_0, x_1, \dots, x_n)$ and by \tilde{p} the homogenization of a singleton or tuple of polynomials p . Let \mathcal{U}_J be the projective variety in \mathbb{P}^n (cf. [19]) defined as

$$(A.2) \quad \text{rank } Jac(\tilde{f}, \tilde{h}, \tilde{g}_J)|_{\tilde{x}} \leq m_1 + |J|, \quad \tilde{h}(\tilde{x}) = \tilde{g}_J(\tilde{x}) = 0.$$

Clearly, if $u \in \mathcal{V}_J$, then $(1, u) \in \mathcal{U}_J$. Suppose otherwise \mathcal{V}_J is infinite, then \mathcal{U}_J is positively dimensional. By Bezout's Theorem (cf. [19]), \mathcal{U}_J must intersect the hyperplane $x_0 = 0$ in \mathbb{P}^n , i.e., (A.2) has a solution like $(0, v)$ with $0 \neq v \in \mathbb{C}^n$. So, v is a solution to the homogeneous polynomial system

$$(A.3) \quad \text{rank } Jac(f^{hom}, h^{hom}, g_J^{hom})|_x \leq m_1 + |J|, \quad h^{hom}(x) = g_J^{hom}(x) = 0.$$

Since $\Delta(h^{hom}, g_J^{hom}) \neq 0$, $\text{rank } Jac(h^{hom}, g_J^{hom})|_x = m_1 + |J|$ for all $0 \neq x \in V_{\mathbb{C}}(h^{hom}, g_J^{hom})$. The rank condition in (A.3) implies that $f^{hom}(v) = 0$ (cf. [30, Section 3]). Hence, v is a nonzero singular solution to

$$f^{hom}(x) = h^{hom}(x) = g_J^{hom}(x) = 0,$$

which contradicts $\Delta(f^{hom}, h^{hom}, g_J^{hom}) \neq 0$ (cf. [30]). So, \mathcal{V}_J must be finite. \square

REFERENCES

- [1] C. Bayer and J. Teichmann. The proof of Tchakaloff's Theorem, *Proc. Amer. Math. Soc.*, 134(2006), pp. 3035-3040.
- [2] A. Ben-Tal and A. Nemirovski. *Lectures on Modern Convex Optimization: Analysis, Algorithms, and Engineering Applications*. MPS-SIAM Series on Optimization, SIAM, Philadelphia, 2001.
- [3] F. Barioli, A. Berman. The maximal CP-rank of rank k completely positive matrices. *Linear Algebra Appl.* 363(2003), pp. 17-33.
- [4] A. Berman and U. Rothblum. A note on the computation of the CP-rank. *Linear Algebra and its Applications* 419 (2006), pp. 1-7.
- [5] A. Berman and N. Shaked-Monderer. *Completely Positive Matrices*, World Scientific, 2003.
- [6] S. Burer. On the copositive representation of binary and continuous nonconvex quadratic programs. *Mathematical Programming*, Ser. A, 120(2009), pp. 479-495.
- [7] J. B. Conway. *A course in Functional Analysis*. Springer-Verlag, 1990 2nd edition.
- [8] R. Curto and L. Fialkow, Recursiveness, positivity, and truncated moment problems. *Houston J. Math.* 17(1991), pp. 603-635.
- [9] R. Curto and L. Fialkow. Solution of the truncated complex moment problem for flat data. *Memoirs of the American Mathematical Society*, 119(1996), No. 568, Amer. Math. Soc., Providence, RI, 1996.
- [10] R. Curto and L. Fialkow. Flat extensions of positive moment matrices: Relations in analytic or conjugate terms. *Operator Th.: Adv. Appl.* 104(1998), pp. 59-82.
- [11] R. Curto and L. Fialkow. Truncated K-moment problems in several variables. *Journal of Operator Theory*, 54(2005), pp. 189-226.
- [12] R. Curto and L. Fialkow. An analogue of the Riesz-Haviland Theorem for the truncated moment problem, *J. Functional Analysis*, 225(2008), 2709-2731.
- [13] E. de Klerk and D.V. Pasechnik. Approximating of the stability number of a graph via copositive programming. *SIAM Journal on Optimization*, 12(4), 875-892.
- [14] P.J. Dickinson and L. Gijben. On the computational complexity of membership problems for the completely positive cone and its dual. *Preprint*, 2011.
- [15] P.J. Dickinson and M. Dür. Linear-time complete positivity detection and decomposition of sparse matrices. *SIAM Journal On Matrix Analysis and Applications*, 33 (2012), No. 3, 701-720.
- [16] M. Dür. Copositive Programming - a Survey. In: M. Diehl, F. Glineur, E. Jarlebring, W. Michiels (Eds.), *Recent Advances in Optimization and its Applications in Engineering*, Springer 2010, pp. 3-20.
- [17] L. Fialkow and J. Nie. Positivity of Riesz functionals and solutions of quadratic and quartic moment problems, *J. Functional Analysis*, 258(2010), no. 1, pp. 328-356.
- [18] L. Fialkow and J. Nie. The truncated moment problem via homogenization and flat extensions. *J. Functional Analysis* 263(2012), no. 6, pp. 1682-1700.
- [19] J. Harris. *Algebraic Geometry, A First Course*. Springer Verlag, 1992.
- [20] J.W. Helton and J. Nie. A semidefinite approach for truncated K-moment problems. *Foundations of Computational Mathematics*, to appear.
- [21] D. Henrion and J. Lasserre. Detecting global optimality and extracting solutions in GloptiPoly. *Positive polynomials in control*, 293-310, Lecture Notes in Control and Inform. Sci., 312, Springer, Berlin, 2005.
- [22] D. Henrion, J. Lasserre and J. Loeferberg. *GloptiPoly 3: moments, optimization and semidefinite programming*. *Optimization Methods and Software* 24(2009), no. 4-5, pp. 761-779.
- [23] J. B. Lasserre. Global optimization with polynomials and the problem of moments. *SIAM J. Optim.* 11(2001), no.3, pp. 796-817.
- [24] J.B. Lasserre. A semidefinite programming approach to the generalized problem of moments. *Mathematical Programming* 112 (2008), pp. 65-92.
- [25] J.B. Lasserre. *Moments, Positive Polynomials and Their Applications*, Imperial College Press, 2009.
- [26] M. Laurent. Revisiting two theorems of Curto and Fialkow on moment matrices. *Proceedings of the American Mathematical Society* 133(2005), no. 10, pp. 2965-2976.

- [27] M. Laurent. Sums of squares, moment matrices and optimization over polynomials, Emerging Applications of Algebraic Geometry, Vol. 149 of IMA Volumes in Mathematics and its Applications, M. Putinar and S. Sullivant (eds), Springer, pages 157-270, 2009.
- [28] Y. Li, A. Kummert, A. Frommer. A linear programming based analysis of the CP-rank of completely positive matrices. *Int. J. Appl. Math. Compu. Sci.* 14 (2004), pp. 25-31.
- [29] M. Putinar. Positive polynomials on compact semi-algebraic sets, *Ind. Univ. Math. J.* 42(1993), pp. 969-984.
- [30] J. Nie. Discriminants and nonnegative polynomials. *Journal of Symbolic Computation* 47(2012), no. 2, pp. 167-191.
- [31] J. Nie. Certifying convergence of Lasserre's hierarchy via flat truncation. *Mathematical Programming*, to appear.
- [32] J. Nie. Optimality conditions and finite convergence of Lasserre's hierarchy. *Preprint*, 2012.
- [33] J. Renegar. On the computational complexity and geometry of the first-order theory of the reals. Parts I, II, III. *J. Symbolic Comput.* 13 (1992), no. 3, 255-352.
- [34] B. Reznick. Some concrete aspects of Hilbert's 17th problem. In *Contemp. Math.*, Vol. 253, pp. 251-272. American Mathematical Society, 2000.
- [35] B. Reznick. Sums of even powers of real linear forms. *Mem. Amer. Math. Soc.*, Vol. 96, No. 463, 1992.
- [36] R. Schneider. *Convex bodies: the Brunn-Minkowski theory*. Encyclopedia of Mathematics and its Applications, Vol. 44. Cambridge University Press, Cambridge, 1993.
- [37] J.F. Sturm. SeDuMi 1.02: A MATLAB toolbox for optimization over symmetric cones. *Optimization Methods and Software*, 11 & 12 (1999), pp. 625-653.
- [38] B. Sturmfels. *Solving systems of polynomial equations*. CBMS Regional Conference Series in Mathematics, 97. American Mathematical Society, Providence, RI, 2002.
- [39] V. Tchakaloff. Formules de cubatures mécanique à coefficients non négatifs. *Bull. Sci. Math.* (2) 81(1957), pp. 123-134.
- [40] M. Todd. Semidefinite optimization. *Acta Numerica* 10(2001), pp. 515-560.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA SAN DIEGO, 9500 GILMAN DRIVE,
LA JOLLA, CA 92093, USA.

E-mail address: njw@math.ucsd.edu